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Consistent shakedown theorems for materials with temperature dependent yield functions

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Abstract

The (elastic) shakedown problem for structures subjected to loads and temperature variations is addressed in the hypothesis of elastic–plastic rate-independent associative material models with temperature-dependent yield functions. Assuming the yield functions convex in the stress/temperature space, a thermodynamically consistent small-deformation thermo-plasticity theory is provided, in which the set of state and evolutive variables includes the temperature and the plastic entropy rate. Within the latter theory the known static (Prager's) and kinematic (König's) shakedown theorems — which hold for yield functions convex in the stress space — are restated in an appropriate consistent format. In contrast with the above known theorems, the restated theorems provide dual lower and upper bound statements for the shakedown limit loads; additionally, the latter theorems can be expressed in terms of only dominant thermo-mechanical loads (generally the vertices of a polyhedral load domain in which the loadings are allowed to range). The shakedown limit load evaluation problem is discussed together with the related shakedown limit state of the structure. A few numerical applications are presented. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The extension of the classical shakedown theorems to thermal loadings and materials with temperature-dependent yield stress was achieved by Prager (1956) for Melan's static theorem, and by König (1982a, 1982b) for Koiter's kinematic theorem — though the latter author quotes Rozemblum (1965) as a previous contributor to this matter. Both extensions considered perfectly plastic materials with yield functions convex in the $\boldsymbol{\sigma}$ stress space for every temperature θ. In contrast with classical

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shakedown theorems, related to temperature-independent yield stress, these two extended theorems exhibit the following undesirable drawbacks:

1. In general, except in the case of linearly dependent yield-stress, they do not constitute dual lower and upper bound theorems for the shakedown limit load whenever the temperature variations are nonstationary and thus (possibly combined with the mechanical loads) are susceptible to amplification.
2. In general, except in case of fully convex yield functions, they cannot be simplified by considering, in place of all load/temperature combinations, only the discrete set of dominant loads; that is, the loading conditions represented by the vertices of the polyhedral load/temperature domain within which the loads and temperature are allowed to range.

It is the purpose of the present paper to show that, if the yield function is convex in the stress/temperature space, the two shakedown theorems can be cast in consistent forms, i.e. free from the above drawbacks, like the classical shakedown theorems. To this purpose, within the framework of thermodynamics of internal variables, suitable thermo-plasticity flow laws will be introduced in which the temperature and the 'plastic entropy' rate will play the role of further state and evolutive variables, respectively.

From experimental data (König, 1987; Hansen and Schreyer, 1994), the yield stress $\kappa = \kappa(\theta)$ is known to decrease with increasing temperature θ and to exhibit a concave shape for a rather wide range of θ (approximately, $-10°C < \theta < 600°C$ for many metal and alloys). It must be remarked that in order to maintain the problem inside the standard thermo-mechanical formulations, the maximum admissible temperature that the material can suffer is always well below the melting temperature, typically no more than one-third of it (Lemaitre and Chaboche, 1985). For modest temperature variations, the hypothesis of temperature-independent, or linearly temperature-dependent, yield stress is usually considered adequate. For more important temperature variations, the above mentioned hypothesis may be the source of unacceptable errors in elastic–plastic analysis, and it then may be more appropriate to assume $\kappa(\theta)$ to be concave, hence $f(\boldsymbol{\sigma}, \theta) = \phi(\boldsymbol{\sigma}) - \kappa(\theta)$ is convex in the $(\boldsymbol{\sigma}, \theta)$ space. Considering that the yield stress $\kappa(\theta)$ is concave for many structural materials, the study of shakedown theory under the general assumption of yield function being convex in the stress/temperature space turns out to be of interest both from the theoretical and practical viewpoints.

The plan of the paper is as follows. In Section 2, the two mentioned extended shakedown theorems are reviewed and their drawbacks discussed. Section 3 is devoted to the thermo-plastic internal-variable material model and to the related associative thermo-plastic flow laws which are expressed in terms of plastic strain rates, kinematic internal variables rates and plastic entropy rate. The thermodynamic consistency of these flow laws is partially discussed in Section 3, but mainly in Section 4, where arguments of the thermodynamics of internal variables are developed with the conclusion that, at least within small-deformation and quasi-static processes, all coupling thermo-mechanical effects can be disregarded. The shakedown problem is then posed in Section 5 and the related (consistent) static and kinematic theorems are addressed in Sections 6 and 7, respectively, showing their differences with respect the existing ones, as well as the cases in which these differences disappear. Section 8 is devoted to the analysis of the shakedown limit state produced by the shakedown limit load, for which static and kinematic approaches are presented. A few numerical applications are presented in Section 9. The conclusion are drawn in Section 10.

2. Review of existing shakedown theorems

For later use, the two extended shakedown theorems mentioned in Section 1 are reported hereafter with reference to a continuous solid body of elastic, perfectly plastic material, which is characterized by a yield function $f(\boldsymbol{\sigma}, \theta) \leq 0$ being convex in the $\boldsymbol{\sigma}$ stress space for every θ value, the body being subjected to quasi-static loads and temperature variations. The plain word 'shakedown' is used to mean elastic shakedown; that is, to mean that the body will eventually respond elastically to any subsequent loads and temperature variations within the given load/temperature domain, after a transient elastic–plastic phase during which some limited amount of plastic strain has been produced (see, e.g. Koiter, 1960; König, 1987; Gokhfeld and Cherniavsky, 1980; Halphen, 1979).

2.1. Static shakedown theorem

Shakedown occurs if, and only if, there exists some time-independent self-stress field, $\hat{\boldsymbol{\sigma}}^R$, such that the stresses resulting from the superposition of this $\hat{\boldsymbol{\sigma}}^R$ with the thermo-elastic stress response, $\boldsymbol{\sigma}^E$, to the loads and temperature variations in any potentially active load/temperature path, nowhere violates the temperature-dependent yield condition, i.e.

$$f(\boldsymbol{\sigma}^E + \hat{\boldsymbol{\sigma}}^R, \theta) \leq 0 \text{ in } V, \forall t \in (0, t_\mathrm{f}), \tag{1}$$

where V denotes the body's domain, t the ordering time-like parameter along an arbitrary load/temperature path, $0 \leq t \leq t_\mathrm{f}$, in the given load/temperature domain, $t_\mathrm{f} > 0$ is any subsequent time

2.2. Kinematic shakedown theorem

Shakedown occurs if, and only if, for any potentially active load/temperature path, $0 \leq t \leq t_\mathrm{f}$, the inequality

$$\int_V \int_0^{t_\mathrm{f}} \left[D(\dot{\boldsymbol{\varepsilon}}^{\mathrm{pc}}, \theta) - \boldsymbol{\sigma}^E : \dot{\boldsymbol{\varepsilon}}^{\mathrm{pc}} \right] \mathrm{d}t \, \mathrm{d}V \geq 0 \tag{2a}$$

is satisfied with arbitrary kinematically admissible (k.a.) plastic strain rate cycles, $\dot{\boldsymbol{\varepsilon}}^{\mathrm{pc}}$, that is, resulting into a self-compatible field $\Delta\boldsymbol{\varepsilon}^{\mathrm{pc}}$, i.e.

$$\Delta\boldsymbol{\varepsilon}^{\mathrm{pc}} := \int_0^{t_\mathrm{f}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{pc}} \, \mathrm{d}t = \nabla^\mathrm{s} \mathbf{u}^c \text{ in } V, \quad \mathbf{u}^c = \mathbf{0} \text{ on } S_u. \tag{2b}$$

The symbol $:=$ means equality by definition, D is the temperature-dependent dissipation function (related to f, hence convex for every θ value), ∇^s is the symmetric part of the gradient operator ∇, $\mathbf{u}^c$ is a time-independent displacement field, S_u is the part of the boundary surface $S = \partial V$, where the displacements are specified.

The above theorems exhibit the drawbacks described in points 1. and 2. of Section 1. The latter drawback is quite obvious, but the former needs further explanation. To this purpose, let the loads and the temperature variations be specified to within a scalar-valued multiplier $\beta > 0$, such that $\boldsymbol{\sigma}^E = \beta\bar{\boldsymbol{\sigma}}^E$ and $\theta = \beta\bar{\theta}$, where $\bar{\boldsymbol{\sigma}}^E$ is the thermo-elastic stress response to the reference loads and temperature variations $\bar{\theta}$. The shakedown safety factor, β_{sh}, can thus be computed as the maximum multiplier β for which shakedown occurs; that is, for which the following conditions hold:

- The static-type condition

$$f(\beta\bar{\boldsymbol{\sigma}}^E + \hat{\boldsymbol{\sigma}}^R, \beta\bar{\theta}) \leq 0 \text{ in } V, \forall t \in (0, t_\mathrm{f}), \tag{3}$$

to be satisfied for every potentially active load/temperature path and some time-independent self-stress field, $\hat{\boldsymbol{\sigma}}^R$;

- The kinematic-type condition

$$\int_V \int_0^{t_f} [D(\dot{\boldsymbol{\varepsilon}}^{pc}, \beta\bar{\theta}) - \beta\bar{\boldsymbol{\sigma}}^E:\dot{\boldsymbol{\varepsilon}}^{pc}] \, dt \, dV \geq 0, \tag{4}$$

to be satisfied for every potentially active load/temperature path and for every k.a. plastic strain rate cycle, $\dot{\boldsymbol{\varepsilon}}^{pc}$.

Obviously, any β complying with the static-type condition (3) is a lower bound to β_{sh}. On the contrary, the kinematic-type condition (4) which holds for any $\beta \leq \beta_{sh}$, rewritten for $\beta = \beta_{sh}$ and considering only those nontrivial k.a. plastic strain rate cycle for which the reference external work is positive, i.e.

$$\int_V \int_0^{t_f} \bar{\boldsymbol{\sigma}}^E:\dot{\boldsymbol{\varepsilon}}^{pc} \, dt \, dV > 0 \tag{5}$$

provides the inequality

$$\beta_{sh} \leq \int_V \int_0^{t_f} D\big(\dot{\boldsymbol{\varepsilon}}^{pc}, \beta_{sh}\bar{\theta}\big) \, dt \, dV \bigg/ \int_V \int_0^{t_f} \bar{\boldsymbol{\sigma}}^E:\dot{\boldsymbol{\varepsilon}}^{pc} \, dt \, dV. \tag{6}$$

This inequality is not a proper upper bound to β_{sh}.

An exception to the latter result occurs for materials with a yield function of the form $f = \phi(\boldsymbol{\sigma}) - \kappa_0(1 - c\theta) \leq 0$, where $\phi(\boldsymbol{\sigma})$ is convex and homogeneous to degree one, κ_0 is the yield stress at the reference temperature $\theta = 0$ and $c > 0$ is some material constant. In this case, it is $D = (1 - c\theta)D_0(\dot{\boldsymbol{\varepsilon}}^p)$ where $D_0(\dot{\boldsymbol{\varepsilon}}^p)$ denotes the dissipation function at $\theta = 0$ (König, 1982a, 1982b), and Eq. (4) can then be transformed as

$$\int_V \int_0^{t_f} \big\{D_0(\dot{\boldsymbol{\varepsilon}}^{pc}) - \beta\big[\bar{\boldsymbol{\sigma}}^E:\dot{\boldsymbol{\varepsilon}}^{pc} + c\bar{\theta}\, D_0(\dot{\boldsymbol{\varepsilon}}^{pc})\big]\big\} \, dt \, dV \geq 0, \tag{7}$$

which, when written for $\beta = \beta_{sh}$, is able to provide proper upper bounds to β_{sh}. Additionally, f being linearly dependent on θ and thus convex also with respect to θ, it can be stated that none of the drawbacks mentioned in Section 1 arise for the considered class of materials. However, this is not the case in the general case of materials with yield functions being nonconvex with respect to the temperature. In the following, the above shakedown theorems will be suitably restated such as not to exhibit the named drawbacks.

3. Inelastic material behavior

A rate-independent associative material model is considered, with a yield function $f(\boldsymbol{\sigma}, \boldsymbol{\chi}, \theta) \leq 0$, where $\boldsymbol{\chi}$ denotes the dual (static) internal variables, f is, by hypothesis, smooth and convex in the space of all its arguments and $\theta = 0$ denotes a reference ambient temperature. Viscous effects are disregarded for simplicity's sake, which is reasonable for temperatures below certain limits. The material behavior is described in the space of the state variables $(\boldsymbol{\sigma}, \boldsymbol{\chi}, \theta)$ by expressing the evolutive laws, or thermo-plastic yielding laws, in analogy with the analogous laws for isothermal process (see e.g. Lubliner, 1990), as follows:

$$\dot{\boldsymbol{\varepsilon}}^{p} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}, \; -\dot{\boldsymbol{\xi}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\chi}}, \; \dot{\boldsymbol{\eta}}^{p} = \dot{\lambda}\frac{\partial f}{\partial \theta}, \tag{8a}$$

$$f(\boldsymbol{\sigma}, \boldsymbol{\chi}, \theta) \leq 0, \; \dot{\lambda} \geq 0. \quad \dot{\lambda} f(\boldsymbol{\sigma}, \boldsymbol{\chi}, \theta) = 0. \tag{8b}$$

Here, $\dot{\lambda}$ is the consistency (or plastic activation) coefficient accounting for the loading/unloading rule through Eqs. (8b). The instantaneous deformation mechanism is described by the set of evolutive variables, $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}, \dot{\eta}^p)$, including the plastic strain rates, $\dot{\boldsymbol{\varepsilon}}^p$, the (kinematic) internal variable rates, $\dot{\boldsymbol{\xi}}$ and the *plastic entropy* rate density, $\dot{\eta}^p$. The energy dissipated per unit volume reads

$$D := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \boldsymbol{\chi} \cdot \dot{\boldsymbol{\xi}} + \theta \dot{\eta}^p \geq 0, \tag{9}$$

which is the *intrinsic thermo-mechanical dissipation* density.

Note that provided f is nonlineraly dependent on $\boldsymbol{\sigma}$ and θ, Eqs. (8a) and (8b) can be uniquely solved with respect to the state variables $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$ and θ, expressing them in terms of a given nontrivial mechanism $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}, \dot{\eta}^p)$, such that D in Eq. (9) turns out to be a one-degree homogeneous function of the evolutive variables and to possess, for any nontrivial mechanism, the following properties:

$$\boldsymbol{\sigma} = \frac{\partial D}{\partial \dot{\boldsymbol{\varepsilon}}^p}, \quad \boldsymbol{\chi} = -\frac{\partial D}{\partial \dot{\boldsymbol{\xi}}}, \quad \theta = \frac{\partial D}{\partial \dot{\eta}^p}. \tag{10}$$

Due to the convexity of f, the following Drucker-type (Drucker, 1960) inequality holds, i.e.

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) : \dot{\boldsymbol{\varepsilon}}^p - (\boldsymbol{\chi} - \boldsymbol{\chi}^*) \cdot \dot{\boldsymbol{\xi}} + (\theta - \theta^*) \dot{\eta}^p \geq 0, \tag{11}$$

for any sets $(\boldsymbol{\sigma}, \boldsymbol{\chi}, \theta)$ and $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}, \dot{\eta}^p)$ corresponding to each other through the constitutive equations (Eqs. (8a) and (8b)), as well as for any, plastically admissible set $(\boldsymbol{\sigma}^*, \boldsymbol{\chi}^*, \theta^*)$, i.e. such that $f(\boldsymbol{\sigma}^*, \boldsymbol{\chi}^*, \theta^*) \leq 0$. The equality sign holds in Eq. (11) if, and only if, $\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$, $\dot{\boldsymbol{\xi}} = \mathbf{0}$ $\dot{\eta}^p = 0$ (in which case, $\boldsymbol{\sigma}^*$, $\boldsymbol{\chi}^*$ and θ^*, may differ from $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$ and θ, and then the mechanism $\dot{\boldsymbol{\varepsilon}}^p$, $\dot{\boldsymbol{\xi}}$, $\dot{\eta}^p$ may be nontrivial).

Eq. (11) is equivalent to a *Maximum Intrinsic Thermoplastic Dissipation Theorem*, extension to the present contest of the classical one (Hill, 1950); that is:

$$D(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}, \dot{\eta}^p) = \max_{(\sigma, \chi, \theta)} \; \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \boldsymbol{\chi} \cdot \dot{\boldsymbol{\xi}} + \theta \dot{\eta}^p$$

subject to:

$$f(\boldsymbol{\sigma}, \boldsymbol{\chi}, \theta) \leq 0, \quad (\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}, \dot{\eta}^p \text{ fixed}). \tag{12}$$

It can be shown that the Kuhn–Tucker conditions of the maximum problem in Eq. (12) coincide with Eqs. (8a) and (8b) and are not only necessary, but also sufficient conditions, but this point is skipped for brevity.

It can be easily checked that the nonnegativity of D in Eq. (9) is always guaranteed. In fact, using Eq. (8a), we can write:

$$D = \dot{\lambda}\left[\boldsymbol{\sigma} : \frac{\partial f}{\partial \boldsymbol{\sigma}} + \boldsymbol{\chi} \cdot \frac{\partial f}{\partial \boldsymbol{\chi}} + \theta \frac{\partial f}{\partial \theta}\right] \geq \dot{\lambda} f(\boldsymbol{\sigma}, \boldsymbol{\chi}, \theta), \tag{13}$$

where the inequality

$$\boldsymbol{\sigma}:\frac{\partial f}{\partial \boldsymbol{\sigma}} + \boldsymbol{\chi}\cdot\frac{\partial f}{\partial \boldsymbol{\chi}} + \theta\frac{\partial f}{\partial \theta} \geq f(\boldsymbol{\sigma}, \boldsymbol{\chi}, \theta) \tag{14}$$

due to the convexity of f has been used. Finally, by the complementarity condition $\dot{\lambda}f = 0$ in Eq. (8b), we have $D \geq 0$ for any arbitrary deformation mechanism.

The consistency of the above constitutive equations from the thermodynamics point of view deserves further discussion, which will be done in the next section. This is devoted to the thermodynamic aspects of the material constitutive behavior and to a possible justification of the nonclassical concept of 'plastic entropy', which appeared in the literature quite recently (Simo and Miehe, 1992; Pantuso and Bathe, 1997; Svedberg and Runesson, 1997; Borino and Polizzotto, 1997a, 1997b). The main point, as shown in detail in the next section, is that the entropy production is sum of two contributions, one related to the independent state variable, i.e. the 'reversible' entropy η^{e}, which appear in the Helmholtz free energy function and then in the state laws, the other, the plastic entropy η^{p} is related to the yield stress variation through the temperature, the rate of this last term multiplied by its dual variable temperature gives a contribution to the thermo-mechanical dissipation function.

4. Thermodynamic considerations

Let the material of Section 3 be further considered in the context of small deformations and let the existence of a specific internal energy, $u = u(\boldsymbol{\varepsilon}^{\mathrm{e}}, \eta^{\mathrm{e}}, \boldsymbol{\xi})$, be postulated. u depends on the elastic strains, $\boldsymbol{\varepsilon}^{\mathrm{e}}$, the 'reversible' entropy, η^{e}, (i.e. the entropy entering the state equations) and the internal variables, $\boldsymbol{\xi}$. The total entropy, in fact, is assumed to be composed of two contributions, that is

$$\dot{\eta} = \dot{\eta}^{\mathrm{e}} + \frac{\theta}{\rho T}\dot{\eta}^{\mathrm{p}}, \tag{15}$$

where $T = T_0 + \theta$ is the absolute temperature, ρ is the mass density and η^{p} is the 'plastic entropy' entering the thermo-plastic constitutive equations. Note that $\dot{\eta}^{\mathrm{p}}$ is multiplied by the ratio $\theta/\rho T$ in Eq. (15) in order to transform $\dot{\eta}^{\mathrm{p}}$ (which is referred to the relative temperature θ and the unit volume) into the 'irreversible' entropy $\dot{\eta}^{i} = \theta\dot{\eta}^{\mathrm{p}}/\rho T$, which, like $\dot{\eta}^{\mathrm{e}}$, is referred to the absolute temperature T and the unit mass. The first thermodynamics principle reads:

$$\rho\dot{u} = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} + r - \operatorname{div}\mathbf{q}, \tag{16}$$

where $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ is the total strain rate, r the heat source per unit volume and $\mathbf{q}$ the heat flux per unit area (see, e.g. Germain et al., 1983; Lemaitre and Chaboche, 1985).

Introducing the Helmholtz free energy, $\psi = \psi(\boldsymbol{\varepsilon}^{\mathrm{e}}, T, \boldsymbol{\xi})$, related to u by $\psi = u - T\eta^{\mathrm{e}}$, as well as the *external entropy production*, $\dot{\eta}_{\mathrm{ext}}$, i.e.

$$\dot{\eta}_{\mathrm{ext}} := \frac{1}{\rho}\left[\frac{r}{T} - \operatorname{div}\left(\frac{\mathbf{q}}{T}\right)\right], \tag{17}$$

we can rearrange Eq. (16) to express the *internal entropy production*, i.e. $\dot{\eta}_{\mathrm{int}} = \dot{\eta} - \dot{\eta}_{\mathrm{ext}}$, as follows

$$\rho T\dot{\eta}_{\mathrm{int}} = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \rho\dot{\psi} - \rho\eta^{\mathrm{e}}\dot{T} + \theta\dot{\eta}^{\mathrm{p}} - \frac{\mathbf{q}}{T}\cdot\nabla T \geq 0, \tag{18}$$

where the nonnegativity sign is a consequence of the second thermodynamics principle.

Eq. (18) is equivalent to the Clausius–Duhem inequality, with which it coincides for $\dot{\eta}^{\mathrm{p}}=0$. Developing the time derivative $\dot{\psi}$, Eq. (18) takes on the form:

$$\rho T\dot{\eta}_{\mathrm{int}}=\left(\boldsymbol{\sigma}-\rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^{\mathrm{e}}}\right):\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}-\rho\left(\eta^{\mathrm{e}}+\frac{\partial\psi}{\partial T}\right)\dot{T}+\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}-\rho\frac{\partial\psi}{\partial\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}+\theta\dot{\eta}^{\mathrm{p}}-\frac{\mathbf{q}}{T}\cdot\nabla T\geq 0. \tag{19}$$

Since the latter inequality holds for any type of deformation and heat flow mechanisms, a classical reasoning enables us to write the relevant state equations, i.e.

$$\boldsymbol{\sigma}=\rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^{\mathrm{e}}},\quad \eta^{\mathrm{e}}=-\frac{\partial\psi}{\partial T},\quad \boldsymbol{\chi}:=\rho\frac{\partial\psi}{\partial\boldsymbol{\xi}} \tag{20}$$

(the third of which is a mere definition), as well as the relation

$$\rho T\dot{\eta}_{\mathrm{int}}=\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}+\theta\dot{\eta}^{\mathrm{p}}-\boldsymbol{\chi}\cdot\dot{\boldsymbol{\xi}}-\frac{\mathbf{q}}{T}\cdot\nabla T=D+D^{\theta}\geq 0, \tag{21}$$

where $D\geq 0$ is recognized as the intrinsic thermo-mechanical dissipation density of Eq. (9), and

$$D^{\theta}:=-\frac{\mathbf{q}}{T}\cdot\nabla T\geq 0 \tag{22}$$

is the thermal diffusion dissipation density. Eq. (21) conforms to a classical result of thermodynamics; namely, the internal entropy production in any deformation and heat flow mechanisms is determined by the related total dissipation density, including the contribution from the plastic entropy. The total entropy rate, $\dot{\eta}=\dot{\eta}_{\mathrm{int}}+\dot{\eta}_{\mathrm{ext}}$, can be obtained using Eqs. (21) and (17). After some easy mathematics, one can write the *entropy balance equation* as

$$\rho T\dot{\eta}=D+r-\operatorname{div}\mathbf{q}. \tag{23}$$

Let the Helmholtz free energy be assumed in the form:

$$\rho\psi=\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{e}}:\mathbf{E}:\boldsymbol{\varepsilon}^{\mathrm{e}}+\Psi(\boldsymbol{\xi})-\boldsymbol{\varepsilon}^{\mathrm{e}}:\mathbf{b}(T-T_0)-\rho C_{\mathrm{v}}T\left(\ln\frac{T}{T_0}-1\right), \tag{24}$$

where $\mathbf{E}$ is the (constant) elastic moduli positive–definite fourth-order tensor (with its usual symmetries), $\mathbf{b}$ is the thermal moduli second-order tensor, C_{v} is the heat capacity per unit mass at constant volume and finally $\Psi(\boldsymbol{\xi})$ is the hardening potential, by hypothesis smooth and convex. By Eq. (20) we have:

$$\boldsymbol{\sigma}=\mathbf{E}:\boldsymbol{\varepsilon}^{\mathrm{e}}-\mathbf{b}(T-T_0), \tag{25a}$$

$$\boldsymbol{\chi}=\boldsymbol{\chi}(\boldsymbol{\xi}):=\frac{\partial\Psi}{\partial\boldsymbol{\xi}} \tag{25b}$$

and

$$\eta^{\mathrm{e}}=C_{\mathrm{v}}\ln\frac{T}{T_0}+\frac{1}{\rho}\mathbf{b}:\boldsymbol{\varepsilon}^{\mathrm{e}}. \tag{25c}$$

These (considering ρ as being constant in time) can be rewritten in an equivalent rate form, i.e.

$$\dot{\boldsymbol{\sigma}}=\mathbf{E}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}-\mathbf{b}\dot{T} \tag{26a}$$

$$\dot{\boldsymbol{\chi}} = \mathbf{H}(\boldsymbol{\xi}) \cdot \dot{\boldsymbol{\xi}} \tag{26b}$$

and

$$\dot{\eta}^{\mathrm{e}} = C_{\mathrm{v}} \dot{T}/T + (1/\rho)\mathbf{b}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}. \tag{26c}$$

where $\mathbf{H}(\boldsymbol{\xi}) = \partial^2 \Psi/(\partial\boldsymbol{\xi} \otimes \partial\boldsymbol{\xi})$ is the (positive–definite) hardening moduli tensor. At this point, introducing (for simplicity's sake) the Fourier equation for a thermally homogeneous and isotropic medium, i.e. $\mathbf{q} = -k\nabla T$, substituting the latter into Eq. (23), remembering Eq. (15) and using the identity $\dot{\boldsymbol{\varepsilon}} = = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$, we can write the *heat conduction differential equation* as:

$$k\nabla^2 T + r + D_{\mathrm{M}} - T\mathbf{b}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} = \rho C_{\mathrm{v}} \dot{T}, \tag{27}$$

where ∇^2 is the Laplacian differential operator and D_{M} is the *intrinsic mechanical dissipation*, i.e.

$$D_{\mathrm{M}} = D - \theta\dot{\eta}^{\mathrm{p}} = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \boldsymbol{\chi} \cdot \dot{\boldsymbol{\xi}}. \tag{28}$$

Eq. (27) shows that heat conduction is not directly influenced by the plastic entropy rate.

Eq. (27), written for the body's domain V with the appropriate boundary conditions on ∂V, cannot be solved to obtain the temperature field evolution without considering the mechanical structural problem with which it is coupled through the thermo-plastic evolutive variables as well as the elastic strain and temperature rates. However, as usual in the framework of small deformations and quasi-static problems (see, e.g. Boley and Weiner, 1960), all these coupling effects can be disregarded, such that Eq. (27) simplifies considerably, i.e.

$$k\nabla^2\theta + r = 0. \tag{29}$$

Thus, the temperature field can now be evaluated independently of the structural mechanical problem, obtaining $\theta(\mathbf{x}, t)$, which can thus be considered known for the solution of the structural problem. We conclude this section by stating:

1. the thermo-plastic constitutive equations of Eq. (3), devised to cope with materials with temperature-dependent yield functions, exhibit a satisfactory thermodynamic consistency, and
2. the usual procedure by which the temperature field, $\theta(\mathbf{x}, t)$, is viewed as assigned in the framework of quasi-static small-deformation structural problems, can also be adopted in the case of materials obeying the above constitutive equations.

5. The structural shakedown problem

Let the solid body (or structure) of Section 2 be made of a material obeying the constitutive equations of Eq. (3), as well as Eqs. (25a) and (25b) or Eqs. (26a) and (26b), and let it be subjected to loads and temperature variations which vary in time in a quasi-static manner. The loads in general include body forces specified in V and surface forces specified on $S_t \subset S$. Impressed displacements specified on $S_u = S/S_t$ may also be included for greater generality. The thermo-mechanical loadings depend on a set of independent parameters, say $\mathbf{Q} = (\mathbf{Q}^L, \mathbf{Q}^\theta)$, $\mathbf{Q}^L$ for the mechanical loads, $\mathbf{Q}^\theta$ for the thermal loading. $\mathbf{Q}^L$ and $\mathbf{Q}^\theta$ are allowed to vary arbitrarily within the (closed) domains Π^L and Π^θ, respectively, these domains being assumed — without loss in generality — as being convex hyperpolyhedra of, respectively, m^L and m^θ vertices. Thus, $\mathbf{Q}$ is allowed to range within $\Pi = \Pi^L \times \Pi^\theta$, which is a convex hyperpolyhedron with $m = m^L m^\theta$ vertices. The vectors $\mathbf{Q}_i$, all $i \in I(m) := \{1, 2, ..., m\}$, which specify the vertices of Π, are referred to as the *dominant* (or basic) thermo-mechanical loads.

Any $\mathbf{Q}$ inside Π can be represented (Polizzotto et al., 1991) as:

$$\mathbf{Q} = \sum_{i=1}^{m} \gamma_i \mathbf{Q}_i, \tag{30}$$

where the coefficients γ_i must satisfy the *admissibility conditions*

$$\gamma_i \geq 0 \quad \forall i \in I(m) \text{ and } \sum_{i=1}^{m} \gamma_i = 1. \tag{31}$$

If the coefficients γ_i vary in all possible ways complying with Eq. (31), $\mathbf{Q}$ will then describe the entire domain Π. Additionally, if the coefficients γ_i are taken as time functions, i.e. $\gamma_i = \gamma_i(t)$, $t \geq 0$, and satisfy Eq. (31) for every t, then Eq. (30) will generate a load path $\mathbf{Q}(t)$ inside Π, that is a potentially active load path, or *Admissible Load History* (ALH). Let $\boldsymbol{\sigma}^E$ denote the thermo-elastic stress response to $\mathbf{Q}$, that is the (fictitious) elastic response of the body when the existence of the yield surface, $f(\boldsymbol{\sigma}, \boldsymbol{\chi}, \theta) = 0$, is ignored. Denoting by $\boldsymbol{\sigma}_i^E(\mathbf{x})$ the thermo-elastic stress response to the dominant load $\mathbf{Q}_i = (\mathbf{Q}_i^L, \mathbf{Q}_i^\theta)$, with the associated basic temperature field $\theta_i(\mathbf{x})$, we can write

$$\boldsymbol{\sigma}^E(\mathbf{x}, t) = \sum_{i=1}^{m} \gamma_i(t) \boldsymbol{\sigma}_i^E(\mathbf{x}) \tag{32a}$$

and

$$\theta(\mathbf{x}, t) = \sum_{i=1}^{m} \gamma_i(t) \theta_i(\mathbf{x}) \tag{32b}$$

to designate (by Eq. (32a)) the thermo-elastic stress response to the load $\mathbf{Q} = \sum_{i=1}^{m} \gamma_i(t), \mathbf{Q}_i$, which includes the temperature field, $\theta(\mathbf{x}, t)$, of Eq. (32b).

The actual inelastic response of the structure to any assigned ALH $\mathbf{Q}(t)$, $t \geq 0$, can, in principle, be computed using the relevant constitutive equations, as well as the equilibrium and compatibility equations. But the main question of our interest here is whether shakedown can be predicted to occur in the considered loading conditions. The shakedown theorems are criteria to ascertain whether shakedown occurs or not, (see, e.g. Koiter, 1960; König, 1987; Gokhfeld and Cherniavsky, 1980). In the following sections, the classical static and kinematic shakedown theorems will be restated to cope with the present material model. To this purpose, the load description given in the first part of this section, with the representation in Eqs. (32a) and (32b), will be helpful.

6. Consistent static shakedown theorem

For the structure considered in Section 5, the following can be stated:

6.1. Consistent static shakedown theorem

A necessary and sufficient condition in order that shakedown occurs in a structure subjected to thermo-mechanical loads allowed to range within a (convex) polyhedral domain Π with dominant loads

$\mathbf{Q}_i$, $i \in I(m)$, is that there exist a time-independent self-stress field $\hat{\boldsymbol{\sigma}}^R$, and a time-independent dual internal variable field, $\hat{\boldsymbol{\chi}}$, such as to satisfy the conditions:

$$f\left(\boldsymbol{\sigma}_i^E + \hat{\boldsymbol{\sigma}}^R, \hat{\boldsymbol{\chi}}, \theta_i\right) \le 0 \quad \text{in } V, \forall i \in I(m). \tag{33}$$

Proof 1. The proof procedure is similar to that of classical shakedown theory. For the *necessity* part of the proof, by hypothesis, let shakedown occur. This implies that, under any ALH, say $\mathbf{Q}(t)$, $t \ge 0$, plastic strains stop being produced at a certain time t_a, after which the structure responds elastically to the thermo-mechanical loads in Π including the dominant loads. Thus Eq. (33) is satisfied with $\hat{\boldsymbol{\sigma}}^R$ and $\hat{\boldsymbol{\chi}}$ being the actual residual stresses and dual internal variables at $t = t_a$.

The proof of the *sufficiency* proceeds showing that plastic strains must stop being produced after a certain time under whatsoever ALH. To this purpose, by hypothesis, let Eq. (33) be satisfied with some $\hat{\boldsymbol{\sigma}}^R$ and $\hat{\boldsymbol{\chi}}$, and let $\mathbf{Q}(t)$, $t \ge 0$ be any ALH, the latter being obtained from Eq. (30) with the time functions $\gamma_i(t)$ arbitrarily chosen, but Eq. (31) complied with for all $t \ge 0$. Eqs. (32a) and (32b) give the related thermo-elastic stress response $\boldsymbol{\sigma}^E = \boldsymbol{\sigma}^E(\mathbf{x}, t)$ and temperature field $\theta = \theta(\mathbf{x}, t)$. Due to the convexity of f, we have:

$$f(\boldsymbol{\sigma}^E + \hat{\boldsymbol{\sigma}}^R, \hat{\boldsymbol{\chi}}, \theta) \le \sum_{i=1}^{m} \gamma_i(t) f\left(\boldsymbol{\sigma}_i^E + \hat{\boldsymbol{\sigma}}^R, \hat{\boldsymbol{\chi}}, \theta_i\right) \le 0 \quad \text{in } V, \forall t \ge 0. \tag{34}$$

Therefore, denoting by $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\chi}$,..., the actual response to the considered ALH, by Eq. (11) with the positions $\boldsymbol{\sigma}' = \hat{\boldsymbol{\sigma}} := \boldsymbol{\sigma}^E + \hat{\boldsymbol{\sigma}}^R$, $\boldsymbol{\chi}' = \hat{\boldsymbol{\chi}}$ and $\theta' = \theta$, we can write:

$$j := (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - (\boldsymbol{\chi} - \hat{\boldsymbol{\chi}}) \cdot \dot{\boldsymbol{\xi}} \ge 0 \quad \text{in } V, \forall t \ge 0. \tag{35}$$

By Eqs. (26a) and (26b), it is

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\boldsymbol{\varepsilon}}^E - \dot{\boldsymbol{\varepsilon}} - \mathbf{E}^{-1} : (\dot{\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}}) \tag{36a}$$

and

$$(\boldsymbol{\chi} - \hat{\boldsymbol{\chi}}) \cdot \dot{\boldsymbol{\xi}} = \dot{w}_{\mathrm{h}} \tag{36b}$$

where $\dot{\boldsymbol{\varepsilon}}^E$ is the total strain rate related to $\dot{\boldsymbol{\sigma}}^E$ and w_{h} is the function:

$$w_{\mathrm{h}} := \Psi(\boldsymbol{\xi}) - \Psi(\hat{\boldsymbol{\xi}}) - \hat{\boldsymbol{\chi}} \cdot (\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}) \ge 0, \tag{37}$$

where $\hat{\boldsymbol{\xi}}$ denotes the internal variables corresponding to $\hat{\boldsymbol{\chi}}$ through Eq. (25b). The nonnegativity of the function w_{h} — devised by Maier (1987) in the framework of shakedown for nonlinarly hardening material models — stems for the assumed convexity of the hardening potential $\Psi(\boldsymbol{\xi})$. Substituting from Eqs. (36a) and (36b) into Eq. (35) and then integrating the latter over V gives

$$\int_V j \, \mathrm{d}V = \int_V (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^E) \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t} \int_V \left\{ \frac{1}{2} (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \mathbf{E}^{-1} : (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) + w_{\mathrm{h}} \right\} \mathrm{d}V \ge 0, \tag{38}$$

with the equality sign on the r.h. side holding if, and only if, $j = 0$ everywhere in V.

Considering that the first integral on the r.h. side of Eq. (38) vanishes by the virtual work principle, it follows that

$$\frac{d}{dt}\int_V \left\{\frac{1}{2}(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}):\mathbf{E}^{-1}:(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})+w_h\right\} dV \leq 0 \quad \forall t \geq 0; \tag{39}$$

that is, the positive definite function $W(t)$, defined for all $t \geq 0$ by

$$W(t) := \int_V \left\{\frac{1}{2}(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}):\mathbf{E}^{-1}:(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})+w_h\right\} dV\Bigg|_t, \tag{40}$$

turns out to be monotonically decreasing all along the body's inelastic straining process, even if $j \neq 0$ only within a small portion of V. Since $W(t)$ is bounded from below, a time t_1 must exist such that for all $t \geq t_1$ it is $\dot{W}(t) = 0$, hence $j = 0$ everywhere in V, which means that Eq. (35) is satisfied as an equality for $t \geq t_1$, i.e.

$$(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}):\dot{\boldsymbol{\varepsilon}}^p - (\boldsymbol{\chi}-\hat{\boldsymbol{\chi}})\cdot\dot{\boldsymbol{\xi}} = 0 \quad \text{in } V, \forall t \geq t_1. \tag{41}$$

The latter condition can be satisfied only if $\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$ and $\dot{\boldsymbol{\xi}} = \mathbf{0}$, hence $\dot{\eta}^p = 0$, everywhere in V after t_1, i.e. if shakedown occurs. The proof is so completed □.

Remark 1. *The difference between the static theorem presented in this section and the analogous one of Section 2 — apart from the internal variables here included — consists in the time-free discrete form of Eq. (33), where the discrete set of dominant loads is considered in place of all ALHs of Eq. (1). This is rendered possible by the assumed convexity of f.*

Remark 2. *Specifying the loads and the (relative) temperature to within a scalar multiplier $\beta > 0$, such that $\boldsymbol{\sigma}_i^E = \beta\bar{\boldsymbol{\sigma}}_i^E$* and *$\theta_i = \beta\bar{\theta}_i$ in Eq. (33), the shakedown safety factor, β_{sh}, can be obtained as the maximum β value for which shakedown occurs, that is, for which the static-type condition (33) is satisfied.*

Remark 3. *If Π^θ is empty, but there is a stationary temperature field, $\theta(\boldsymbol{x})$,* in combination with the loads *$\boldsymbol{Q}^L \in \Pi^L$, it is $m = m^L$ and Eq. (32a) reads*

$$\boldsymbol{\sigma}^E(\mathbf{x}, t) = \sum_{i=1}^{m}\gamma_i(t)\boldsymbol{\sigma}_i^{EL}(\mathbf{x}) + \boldsymbol{\sigma}^{E\theta}(\mathbf{x}) \tag{42}$$

where $\boldsymbol{\sigma}_i^{EL}$ and *$\boldsymbol{\sigma}^{E\theta}$ denote the elastic stress responses to the loads $\mathbf{Q}_i^L$ and to the temperature field θ, respectively. The static shakedown theorem remains unaltered, but Eq. (33) can more precisely be written as*

$$f(\boldsymbol{\sigma}_i^{EL} + \boldsymbol{\sigma}^{E\theta} + \hat{\boldsymbol{\sigma}}^R, \hat{\boldsymbol{\chi}}, \theta) \leq 0 \quad \text{in } V, \forall i \in I(m). \tag{43}$$

It is easily recognized that the theorem's proof continues to hold, but the convexity of f with respect to θ *is no longer required, as in fact Eq. (34) now takes on the milder form*

$$f(\boldsymbol{\sigma}^{EL} + \boldsymbol{\sigma}^{E\theta} + \hat{\boldsymbol{\sigma}}^{R}, \hat{\boldsymbol{\chi}}, \theta) \leq \sum_{i=1}^{m} \gamma_i(t) f(\boldsymbol{\sigma}_i^{EL} + \boldsymbol{\sigma}^{E\theta} + \hat{\boldsymbol{\sigma}}^{R}, \hat{\boldsymbol{\chi}}, \theta) \leq 0 \quad \text{in } V, \forall t \geq 0. \tag{44}$$

This means that, in the case of stationary thermal loading, the static shakedown theorem of this section coincides with that of Section 2 (straightforwardly generalized to include internal variables).

7. Consistent kinematic shakedown theorem

The kinematic shakedown theorem grounds on the concept of Plastic Accumulation Mechanism (PAM) which generalizes that of kinematically admissible plastic strain rate cycle of Koiter (1960), see Polizzotto et al. (1991). In essence, such a PAM consists of set of (fictitious) plastic strain fields in one-to-one correspondence with the set of dominant loads, and resulting into a cumulated strain field that is self-compatible. In case of polyhedral load domain Π with m dominant loads $\mathbf{Q}_i$, every PAM will consist of m such strain fields.

In the present context, a PAM includes m plastic strain fields, $\boldsymbol{\varepsilon}_i^{\mathrm{pc}}$, m internal variable fields, $\boldsymbol{\xi}_i^c$ and m plastic entropy fields, η_i^{pc}, such as to satisfy the homogeneous compatibility conditions:

$$\sum_{i=1}^{m} \boldsymbol{\varepsilon}^{\mathrm{pc}_i} = \nabla^{\mathrm{s}} \mathbf{u}^c \text{ in } V, \quad \mathbf{u}^c = \mathbf{0} \text{ on } S_u \tag{45a}$$

and

$$\sum_{i=1}^{m} \boldsymbol{\xi}_i^c = \mathbf{0} \text{ in } V. \tag{45b}$$

No constraints are imposed on the η_i^{pc} fields. Let M denote the set of all PAMs (including the trivial one), and let $M^+ \subset M$ be the subset of (nontrivial) PAMs such that the dominant loads $\mathbf{Q}_i$, perform globally positive work through every such PAM. With these definitions in mind, the following can be proved for the structure of Section 5.

7.1. Consistent kinematic shakedown theorem

A necessary and sufficient condition in order that shakedown occurs in a structure subjected to thermo-mechanical loads allowed to range within a (convex) polyhedra domain Π with dominant loads $\mathbf{Q}_i$, $i \in I(m)$, is that the inequality

$$K\left[\boldsymbol{\varepsilon}_i^{\mathrm{pc}}, \boldsymbol{\xi}_i^c, \eta_i^{\mathrm{pc}}\right] := \int_V \sum_{j=1}^{m} \left\{ D\left(\boldsymbol{\varepsilon}_j^{\mathrm{pc}}, \boldsymbol{\xi}_j^c, \eta_j^{\mathrm{pc}}\right) - \left[\boldsymbol{\sigma}_j^E : \boldsymbol{\varepsilon}_j^{\mathrm{pc}} + \theta_j \eta_j^{\mathrm{pc}}\right] \right\} \mathrm{d}V \geq 0 \tag{46}$$

is satisfied for every PAM in M.

Proof 2. The proof proceeds in analogy with Polizzotto et al. (1991), with arguments different from those of Koiter (1960). The *necessity* is first proved and to this purpose shakedown is assumed to occur.

Thus, by the static shakedown theorem of Section 6, there must exist some self-stress field, $\hat{\boldsymbol{\sigma}}^R$, and a dual internal variable field, $\hat{\boldsymbol{\chi}}$, both time-independent, such that Eq. (33) is satisfied. Then, by absurdity, let us assume that Eq. (46) is violated; that is, there exists some PAM, say $\tilde{\boldsymbol{\varepsilon}}_i^{pc}$, $\tilde{\boldsymbol{\xi}}_i^c$, $\tilde{\eta}_i^{pc}$, (i = 1, 2,..., m), such that the functional K in Eq. (46) takes on a negative value, i.e.

$$K\left[\tilde{\boldsymbol{\varepsilon}}_i^{pc}, \tilde{\boldsymbol{\xi}}_i^c, \tilde{\eta}_i^{pc}\right] < 0. \tag{47}$$

Applying the maximum thermo-plastic dissipation theorem (Eq. (11) or Eq. (12)), we can write the inequality:

$$D\left(\tilde{\boldsymbol{\varepsilon}}_i^{pc}, \tilde{\boldsymbol{\xi}}_i^c, \tilde{\eta}_i^{pc}\right) \geq \left(\boldsymbol{\sigma}_i^E + \hat{\boldsymbol{\sigma}}^R\right):\tilde{\boldsymbol{\varepsilon}}_i^{pc} - \hat{\boldsymbol{\chi}}:\tilde{\boldsymbol{\xi}}_i^c + \theta_i \tilde{\eta}_i^{pc} \quad \text{in } V, \forall i \in I(m). \tag{48}$$

This, after integration over V and summing with respect to $i \in I(m)$, transforms into

$$K\left[\tilde{\boldsymbol{\varepsilon}}_i^{pc}, \tilde{\boldsymbol{\xi}}_i^c, \tilde{\eta}_i^{pc}\right] \geq \int_V \hat{\boldsymbol{\sigma}}^R:\sum_{i=1}^m \tilde{\boldsymbol{\varepsilon}}_i^{pc}\, dV - \int_V \hat{\boldsymbol{\chi}} \cdot \sum_{i=1}^m \tilde{\boldsymbol{\xi}}_i^c\, dV, \tag{49}$$

where, by the compatibility conditions (Eqs. (45a) and (45b)) and the virtual work principle, the r.h. side is recognized to vanish and thus Eq. (49) contradicts the initial assumption in Eq. (47). The conclusion is that, if shakedown occurs, Eq. (46) cannot be violated by any PAM whatsoever.

To prove the *sufficiency*, let us assume that Eq. (46) is satisfied for every PAM. This obviously amounts to stating that the minimization problem

$$\min K\left[\boldsymbol{\varepsilon}_i^{pc}, \boldsymbol{\xi}_i^c, \eta_i^{pc}\right] \text{ in the set } M \text{ of PAMs} \tag{50}$$

has an absolute minimum, as in fact K vanishes for a trivial PAM (but, possibly, also for some nontrivial one). Because Eq. (50) admits a solution, a solution must possess also the related Euler–Lagrange equations. The latter equations are easily derived by taking into account the constraints (Eqs. (45a) and (45b)) and writing the relevant augmented Lagrangian functional, i.e.

$$K_a = K\left[\boldsymbol{\varepsilon}_i^{pc}, \boldsymbol{\xi}_i^c, \eta_i^{pc}\right] - \int_V \hat{\boldsymbol{\sigma}}^R:\left(\sum_{i=1}^m \boldsymbol{\varepsilon}_i^{pc} - \nabla^s \mathbf{u}^c\right) dV + \int_{S_u} \mathbf{u}^c \cdot \hat{\boldsymbol{\sigma}}^R \cdot \mathbf{n}\, dS + \int_V \hat{\boldsymbol{\chi}} \cdot \sum_{i=1}^m \boldsymbol{\xi}_i^c\, dV, \tag{51}$$

where $\mathbf{n}$ is the unit external normal to S, whereas $\hat{\boldsymbol{\sigma}}^R(\mathbf{x})$ and $\hat{\boldsymbol{\chi}}(\mathbf{x})$ denote Lagrange multipliers. In the hypothesis that all the field and boundary functions enjoy the necessary continuity requisites to apply the classical procedure of the calculus of variations, the first variation of Eq. (51), after some mathematics including the application of the divergence theorem, reads:

$$\begin{aligned}
\delta K_a = {} & \int_V \sum_{i=1}^m \delta\boldsymbol{\varepsilon}_i^{pc}:\left[\frac{\partial D}{\partial \boldsymbol{\varepsilon}_i^{pc}} - \boldsymbol{\sigma}_i^E - \hat{\boldsymbol{\sigma}}^R\right] dV + \int_V \sum_{i=1}^m \delta\boldsymbol{\xi}_i^c \cdot \left[\frac{\partial D}{\partial \boldsymbol{\xi}_i^c} + \hat{\boldsymbol{\chi}}\right] dV \\
& + \int_V \sum_{i=1}^m \delta\eta_i^{pc}\left[\frac{\partial D}{\partial \eta_i^{pc}} - \theta_i\right] dV - \int_V \delta\mathbf{u}^c \cdot \operatorname{div} \hat{\boldsymbol{\sigma}}^R\, dV + \int_{S_t} \delta\mathbf{u}^c \cdot \hat{\boldsymbol{\sigma}}^R \cdot \mathbf{n}\, dS \\
& - \int_V \delta\hat{\boldsymbol{\sigma}}^R:\left(\sum_{i=1}^m \boldsymbol{\varepsilon}_i^{pc} - \nabla^s \mathbf{u}^c\right) dV + \int_{S_u} \mathbf{u}^c \cdot \delta\hat{\boldsymbol{\sigma}}^R \cdot \mathbf{n}\, dS + \int_V \delta\hat{\boldsymbol{\chi}} \cdot \sum_{i=1}^m \boldsymbol{\xi}_i^c\, dV.
\end{aligned} \tag{52}$$

Therefore, the Euler–Lagrange equations read, beside Eqs. (45a) and (45b):

$$\boldsymbol{\sigma}_i^E + \hat{\boldsymbol{\sigma}}^R = \frac{\partial D}{\partial \boldsymbol{\varepsilon}_i^{\text{pc}}}, \quad \hat{\boldsymbol{\chi}} = -\frac{\partial D}{\partial \boldsymbol{\xi}_i^c}, \quad \theta_i = \frac{\partial D}{\partial \eta_i^{\text{pc}}} \text{ in } V, \forall i \in I(m) \tag{53a}$$

$$\text{div}\, \hat{\boldsymbol{\sigma}}^R = \mathbf{0} \text{ in } V, \quad \hat{\boldsymbol{\sigma}}^R \cdot \mathbf{n} = \mathbf{0} \text{ on } S_t. \tag{53b}$$

In other words, as a consequence of the assumed validity of Eq. (46), there exist some time-independent self stress and dual internal variable fields, namely the Lagrange multipliers $\hat{\boldsymbol{\sigma}}^R$ and $\hat{\boldsymbol{\chi}}$, such that the fields on the l.h. side of Eq. (53a), being derived from the (convex) dissipation function D through (generalized) partial derivatives with respect to the related evolutive variables, are plastically admissible, i.e. Eq. (33) is satisfied. Thus, by the static theorem, shakedown occurs. The proof is so completed □.

Remark 4. *The difference between the theorem presented in this section and the analogous one of Section 2 — apart from the internal variables herein included — is twofold. First because of the presence of the plastic entropy as an additional ingredient; second, because of the time-free discrete form of Eq. (46), where the discrete set of dominant loads are considered instead of all ALHs. Again, this is rendered possible by the convexity of f (and thus of D).*

Remark 5. *Specifying the loads and the temperature variations to within a scalar multiplier $\beta > 0$, such that $\boldsymbol{\sigma}_i^E = \bar{\boldsymbol{\sigma}}_i^E$, $\theta_i = \bar{\theta}_i$ in Eq. (46), the latter equation reads*

$$\int_V \sum_{i=1}^m D\big(\boldsymbol{\varepsilon}_i^{\text{pc}}, \boldsymbol{\xi}_i^c, \eta_i^{\text{pc}}\big)\, \mathrm{d}V \geq \beta \int_V \sum_{i=1}^m \Big[\bar{\boldsymbol{\sigma}}_i^E : \boldsymbol{\varepsilon}_i^{\text{pc}} + \bar{\theta}_i \eta_i^{\text{pc}}\Big]\, \mathrm{d}V \tag{54}$$

which holds for any $\beta \leq \beta_{sh}$. In particular, assuming $\beta = \beta_{sh}$ and considering only PAMs in the subset M^+, i.e. PAMs such that

$$\int_V \sum_{i=1}^m \Big[\bar{\boldsymbol{\sigma}}_i^E : \boldsymbol{\varepsilon}_i^{\text{pc}} + \bar{\theta}_i \eta_i^{\text{pc}}\Big]\, \mathrm{d}V > 0, \tag{55}$$

Eq. (54) can be rewritten as

$$\beta_{\text{sh}} \leq \int_V \sum_{i=1}^m D\big(\boldsymbol{\varepsilon}_i^{\text{pc}}, \boldsymbol{\xi}_i^c, \eta_i^{\text{pc}}\big)\, \mathrm{d}V \bigg/ \int_V \sum_{i=1}^m \Big[\bar{\boldsymbol{\sigma}}_i^E : \boldsymbol{\varepsilon}_i^{\text{pc}} + \bar{\theta}_i \eta_i^{\text{pc}}\Big]\, \mathrm{d}V, \tag{56}$$

which can be used to compute upper bounds to β_{sh}. Obviously, the minimum value of the ratio on the r.h. side of Eq. (56) in the subset M^+ coincides with *β_{sh}, since, otherwise, Eq. (54) would be satisfied with for β values greater than β_{sh}.*

Remark 6. *In the case considered in Remark 3, i.e. the case $\Pi^\theta = \varnothing$, but with a stationary temperature* *field, $\theta(\boldsymbol{x})$, the above kinematic shakedown theorem simplifies. In fact, being $\partial f/\partial\theta \equiv 0$, the plastic entropy rate, $\dot{\eta}^{\mathrm{p}}$, disappears from all equations and Eq. (46) takes on the reduced form:*

$$K[\boldsymbol{\varepsilon}_i^{\mathrm{pc}}, \boldsymbol{\xi}_i^c] := \int_V \sum_{i=1}^{m} \left\{ D\left(\boldsymbol{\varepsilon}_i^{\mathrm{pc}}, \boldsymbol{\xi}_i^c, \theta\right) - \boldsymbol{\sigma}_i^{EL} : \boldsymbol{\varepsilon}_i^{\mathrm{pc}} \right\} \mathrm{d}V \geq 0. \tag{57}$$

This inequality must be satisfied for every PAM, a PAM being defined as a set of m fields ($\boldsymbol{\varepsilon}_i^{\mathrm{pc}}$, $\boldsymbol{\xi}_i^c$) satisfying Eqs. (45a) and (45b). Note that the integral,

$$\int_V \boldsymbol{\sigma}^{E\theta} : \left(\sum_{i=1}^{m} \boldsymbol{\varepsilon}_i^{\mathrm{pc}} \right) \mathrm{d}V = 0, \tag{58}$$

drops from Eq. (46). The theorem proof can proceed in exact the same way as in the general case, but the convexity of f with respect to θ is no longer required, as in fact the third equality of Eq. (53a) now drops and D in Eq. (57) is allowed to be nonconvex with respect to θ. This means that, in the case of stationary thermal loading, the kinematic shakedown theorem here proved, coincides with that of Section 2 (straightforwardly extended to internal variable material models). In the considered case, the thermal loading can be treated as a permanent loading in the search for the shakedown safety factor, β_{sh}, and Eq. (56) reads

$$\beta_{\mathrm{sh}} \leq \int_V \sum_{i=1}^{m} D\left(\boldsymbol{\varepsilon}_i^{\mathrm{pc}}, \boldsymbol{\xi}_i^c, \theta\right) \mathrm{d}V \Bigg/ \int_V \sum_{i=1}^{m} \bar{\boldsymbol{\sigma}}_i^{EL} : \boldsymbol{\varepsilon}_i^{\mathrm{pc}} \, \mathrm{d}V. \tag{59}$$

8. Shakedown limit load and related limit state

In this section, for greater generality, the dominant loads and temperature variations are specified as $\boldsymbol{\sigma}_i^E = \boldsymbol{\sigma}^{EP} + \beta\bar{\boldsymbol{\sigma}}_i^E$ and, $\theta_i = \theta_0 + \beta\bar{\theta}_i$, $\forall i \in I(m)$, where $\boldsymbol{\sigma}^{EP}$ is the elastic stress response to a given steady load $\mathbf{P}$ and $\theta_0 > 0$ is the ambient temperature. For $\beta < \beta_{\mathrm{sh}}$, shakedown occurs and the structure is safe. For $\beta = \beta_{\mathrm{sh}}$, shakedown still occurs, but the structure is in a (shakedown) limit state, characterized by an impending *inadaptation* (or *noninstantaneous plastic*) collapse. The latter exhibits either an incremental (or *ratcheting*) collapse mode (in which the plastic strain increment per cycle is not identically vanishing), or an *alternating plasticity* collapse mode (in which the opposite occurs). Knowing the features of this limit state is paramount in order to assess the kind of collapse mode that takes place in the structure for β slightly exceeding β_{sh}. The equations governing the shakedown limit state can be derived by addressing the problem of evaluating the shakedown limit load, i.e. β_{sh}. This task, already accomplished in the framework of classical shakedown theory by Panzeca and Polizzotto (1988), Polizzotto et al. (1991) and Polizzotto (1993, 1995), will be pursued in this section.

Using the static shakedown theorem of Section 6, β_{sh} can, in principle, be evaluated solving the following problem:

$$\beta_{\mathrm{sh}} = \max_{(\beta,\, \hat{\sigma}^R,\, \hat{\chi})} \beta, \tag{60a}$$

subject to:

$$f\left(\boldsymbol{\sigma}^{EP} + \beta\bar{\boldsymbol{\sigma}}_i^E + \hat{\boldsymbol{\sigma}}^R, \hat{\boldsymbol{\chi}}, \theta_0 + \beta\bar{\theta}_i\right) \leq 0 \quad \text{in } V, \forall i \in I(m), \tag{60b}$$

$$\operatorname{div} \hat{\boldsymbol{\sigma}}^R = 0 \text{ in } V, \qquad \hat{\boldsymbol{\sigma}}^R \cdot \mathbf{n} = \mathbf{0} \text{ on } S_t. \tag{60c}$$

Making use of the classical Lagrange multiplier method, let the augmented Lagrangian function be written as

$$L = -\beta + \int_V \sum_{i=1}^{m} \lambda_i^c f\left(\hat{\boldsymbol{\sigma}}_i, \hat{\boldsymbol{\chi}}, \hat{\theta}_i\right) \mathrm{d}V + \int_V \mathbf{u}^c \cdot \operatorname{div} \hat{\boldsymbol{\sigma}}^R \, \mathrm{d}V - \int_{S_u} \mathbf{u}^c \cdot \hat{\boldsymbol{\sigma}}^R \cdot \mathbf{n} \, \mathrm{d}S, \tag{61}$$

where the positions $\hat{\boldsymbol{\sigma}}_i := \boldsymbol{\sigma}^{EP} + \beta \bar{\boldsymbol{\sigma}}_i^E + \hat{\boldsymbol{\sigma}}^R$ and $\hat{\theta} := \theta_0 + \beta \bar{\theta}_i$ hold, whereas $\lambda_i^c(\mathbf{x}) \geq 0$, $i \in I(m)$, and $\mathbf{u}^c(\mathbf{x})$ are Lagrange multipliers. The first variation of L, after application of the divergence theorem and with some re-ordering, reads:

$$\begin{aligned}
\delta L = {} & \delta\beta \left[-1 + \int_V \sum_{i=1}^{m} \left\{ \bar{\boldsymbol{\sigma}}_i^E : \frac{\partial f}{\partial \hat{\boldsymbol{\sigma}}_i} \lambda_i^c + \bar{\theta}_i \frac{\partial f}{\partial \hat{\theta}_i} \lambda_i^c \right\} \mathrm{d}V \right] + \int_V \delta \hat{\boldsymbol{\sigma}}^R : \left[\sum_{i=1}^{m} \frac{\partial f}{\partial \hat{\boldsymbol{\sigma}}_i} \lambda_i^c - \nabla^s \mathbf{u}^c \right] \mathrm{d}V \\
& + \int_{S_t} \mathbf{u}^c \cdot \delta \hat{\boldsymbol{\sigma}}^R \cdot \mathbf{n} \, \mathrm{d}S + \int_V \delta \hat{\boldsymbol{\chi}}^R \cdot \sum_{i=1}^{m} \frac{\partial f}{\partial \hat{\boldsymbol{\chi}}} \lambda_i^c + \int_V \sum_{i=1}^{m} \delta \lambda_i^c f(\hat{\boldsymbol{\sigma}}_i, \hat{\boldsymbol{\chi}}, \hat{\theta}_i) \, \mathrm{d}V \\
& + \int_V \delta \mathbf{u}^c \cdot \operatorname{div} \hat{\boldsymbol{\sigma}}^R \, \mathrm{d}V - \int_{S_u} \delta \mathbf{u}^c \cdot \hat{\boldsymbol{\sigma}}^R \cdot \mathbf{n} \, \mathrm{d}S.
\end{aligned} \tag{62}$$

Then, the Euler–Lagrange equations of Eqs. (60a), (60b) and (60c) are, besides Eq. (60c), the following

$$f\left(\hat{\boldsymbol{\sigma}}_i, \hat{\boldsymbol{\chi}}, \hat{\theta}_i\right) \leq 0, \quad \lambda_i^c \geq 0, \quad \lambda_i^c f\left(\hat{\boldsymbol{\sigma}}_i, \hat{\boldsymbol{\chi}}, \hat{\theta}_i\right) = 0 \text{ in } V, \forall i \in I(m); \tag{63a}$$

$$\hat{\boldsymbol{\sigma}}_i := \boldsymbol{\sigma}^{EP} + \beta \bar{\boldsymbol{\sigma}}_i^E + \hat{\boldsymbol{\sigma}}^R, \quad \hat{\theta}_i := \theta_0 + \beta \bar{\theta}_i \text{ in } V, \forall i \in I(m); \tag{63b}$$

$$\boldsymbol{\varepsilon}_i^{\mathrm{pc}} := \lambda_i^c \frac{\partial f}{\partial \hat{\boldsymbol{\sigma}}_i}, \quad -\boldsymbol{\xi}_i^c := \lambda_i^c \frac{\partial f}{\partial \hat{\boldsymbol{\chi}}}, \quad \eta_i^{\mathrm{pc}} := \lambda_i^c \frac{\partial f}{\partial \hat{\theta}_i} \text{ in } V, \forall i \in I(m); \tag{63c}$$

$$\Delta \boldsymbol{\varepsilon}^{\mathrm{pc}} \equiv \sum_{i=1}^{m} \boldsymbol{\varepsilon}_i^{\mathrm{pc}} = \nabla^s \mathbf{u}^c \text{ in } V, \quad \mathbf{u}^c = \mathbf{0} \quad \text{on } S_u; \tag{63d}$$

$$\sum_{i=1}^{m} \boldsymbol{\xi}_i^c = \mathbf{0} \quad \text{in } V \tag{63e}$$

and

$$\int_V \sum_{i=1}^{m} \left\{ \bar{\boldsymbol{\sigma}}_i^E : \boldsymbol{\varepsilon}_i^{\mathrm{pc}} + \bar{\theta}_i \eta_i^{\mathrm{pc}} \right\} = 1. \tag{63f}$$

From the latter equations, the meanings of the Lagrange multipliers λ_i^c and $\mathbf{u}^c$ transpire as plastic coefficients and displacements respectively.

Eqs. (63a), (63b), (63c), (63d), (63e) and (63f) — which express necessary conditions — assess, by Eqs. (63d), (63e) and (63f), the existence of a PAM in the subset M^+, complying with the thermo-plastic constitutive flow laws (Eqs. (63a), (63b) and (63c)). This PAM describes the *impending* inadaptation collapse mechanism, which turns out to exhibit a ratchetting mode if $\mathbf{u}^c \not\equiv \mathbf{0}$, or an alternating plasticity collapse mode if $\mathbf{u}^c \equiv \mathbf{0}$ in V. Furthermore the following can be proved:

1. Eqs. (63a), (63b), (63c), (63d), (63e) and (63f) are not only necessary, but also sufficient conditions for the problem expressed by Eqs. (60a), (60b) and (60c). In fact, denoting by starred symbols the solution to Eqs. (63a), (63b), (63c), (63d), (63e) and (63f) and Eq. (60c), on one hand we have that $\hat{\boldsymbol{\sigma}}^{R*}$ and $\hat{\boldsymbol{\chi}}^*$ possess the requisites required by the static shakedown theorem, and thus $\beta^* \le \beta_{\text{sh}}$. On the other hand, the PAM described by Eqs. (63d), (63e) and (63f) can be utilized to compute an upper bound to β_{sh} through Eq. (56), which now reads

$$\beta_{\text{sh}} \le \int_V \sum_{i=1}^m D\left(\boldsymbol{\varepsilon}_i^{\text{pc}^*}, \boldsymbol{\xi}_i^{c^*}, \eta_i^{\text{pc}^*}\right) \mathrm{d}V - \int_V \left[\boldsymbol{\sigma}^{EP} : \Delta\boldsymbol{\varepsilon}^{\text{pc}^*} + \theta_0 \Delta\eta^{\text{pc}^*}\right] \mathrm{d}V, \tag{64}$$

where $\delta\eta^{\text{pc}*} \equiv \sum_{i=1}^m \eta_i^{\text{pc}*}$. But, by Eqs. (63a), (63b) and (63c),

$$D\left(\boldsymbol{\varepsilon}_i^{\text{pc}*}, \boldsymbol{\xi}_i^{c*}, \eta_i^{\text{pc}*}\right) = \left(\boldsymbol{\sigma}^{EP} + \beta^* \bar{\boldsymbol{\sigma}}_i^E + \hat{\boldsymbol{\sigma}}^{R^*}\right) : \boldsymbol{\varepsilon}_i^{\text{pc}*} - \hat{\boldsymbol{\chi}}^* \cdot \boldsymbol{\xi}_i^{c*} + \left(\theta_0 + \beta^* \bar{\theta}_i\right)\eta_i^{\text{pc}*}, \tag{65}$$

which, after integration over V and summing with respect to $i \in I(m)$, reveals that the r.h. side of Eq. (64) equals β^*. Therefore, Eq. (64) reads $\beta_{\text{sh}} \le \beta^*$ and, combined with the previous result, it is necessarily $\beta^* = \beta_{\text{sh}}$.

2. Eqs. (63a), (63b), (63c), (63d), (63e) and (63f) and Eq. (60c) admit a unique solution for all, except possibly for $\hat{\boldsymbol{\sigma}}^R$ and $\hat{\boldsymbol{\chi}}$ in the region(s) of V (if any) where no (fictitious) plastic strains occur. In fact, denoting with symbols as $(\cdot)'$ and $(\cdot)''$ two solutions, which are assumed to exist, and introducing the symbol $\Delta(\cdot) := (\cdot)' - (\cdot)''$, we can write:

$$\Delta\hat{\boldsymbol{\sigma}}_i : \Delta\boldsymbol{\varepsilon}_i^{\text{pc}} - \Delta\hat{\boldsymbol{\chi}} \cdot \Delta\boldsymbol{\xi}_i^c + \Delta\hat{\theta}_i \Delta\eta_i^{\text{pc}} \ge 0 \text{ in } V, \forall i \in I(m). \tag{66}$$

Noting that (being $\beta' = \beta''$) $\Delta\hat{\boldsymbol{\sigma}}_i = \Delta\hat{\boldsymbol{\sigma}}^R$, $\Delta\hat{\theta}_i = 0$ everywhere in V and for all $i \in I(m)$, integration of Eq. (66) over V and summing with respect to $i \in I(m)$ yields:

$$\int_V \Delta\hat{\boldsymbol{\sigma}}^R : \Delta\left(\sum_{i=1}^m \boldsymbol{\varepsilon}_i^{\text{pc}}\right) \mathrm{d}V - \int_V \Delta\hat{\boldsymbol{\chi}} \cdot \Delta\left(\sum_{i=1}^m \boldsymbol{\xi}_i^c\right) \mathrm{d}V \ge 0. \tag{67}$$

Both integrals in Eq. (67) vanishing by Eqs. (63d), (63e) and (60c) and by the virtual work principle, it follows that Eq. (66) is always satisfied as an equality, i.e.

$$\Delta\hat{\boldsymbol{\sigma}}^R : \Delta\boldsymbol{\varepsilon}_i^{\text{pc}} - \Delta\hat{\boldsymbol{\chi}} \cdot \Delta\boldsymbol{\xi}_i^c = 0 \text{ in } V, \forall i \in I(m). \tag{68}$$

This identity (remembering Eq. (11) with its consequences) implies that $\Delta\boldsymbol{\varepsilon}_i^{\text{pc}} \equiv \mathbf{0}$, $\Delta\boldsymbol{\xi}_i^c \equiv \mathbf{0}$, hence $\Delta\eta_i^{\text{pc}} \equiv 0$, i.e. $\boldsymbol{\varepsilon}_i^{\text{pc}\prime} = \boldsymbol{\varepsilon}_i^{\text{pc}\prime\prime}$, $\boldsymbol{\xi}_i^{c\prime} = \boldsymbol{\xi}_i^{c\prime\prime}$ and $\eta_i^{\text{pc}\prime} = \eta_i^{\text{pc}\prime\prime}$ everywhere in V, hence $\mathbf{u}^{c\prime} = \mathbf{u}^{c\prime\prime}$ in V but $\hat{\boldsymbol{\sigma}}^{R\prime}$ and $\hat{\boldsymbol{\chi}}'$ may be different from $\hat{\boldsymbol{\sigma}}^{R\prime\prime}$ and $\hat{\boldsymbol{\chi}}''$, respectively, at points in V where no plastic strains occur in the impending deformation process.

The shakedown safety factor, $\beta_{\rm sh}$, can also be determined making use of the kinematic shakedown theorem of Section 7 and, in particular, of Eq. (56). So we can write the problem:

$$\beta_{\rm sh} = \min_{\left(\varepsilon_i^{\rm pc},\, \xi_i^c,\, \eta_i^{\rm pc},\, u^c\right)} \int_V \left\{ \sum_{i=1}^m D\left(\boldsymbol{\varepsilon}_i^{\rm pc}, \boldsymbol{\xi}_i^c, \eta_i^{\rm pc}\right) - \boldsymbol{\sigma}^{EP}:\Delta\boldsymbol{\varepsilon}^{\rm pc} + \theta_0 \Delta\eta^{\rm pc} \right\} {\rm d}V, \tag{69a}$$

subject to:

$$\int_V \sum_{i=1}^m \left(\bar{\boldsymbol{\sigma}}_i^E:\boldsymbol{\varepsilon}_i^{\rm pc} + \bar{\theta}_i \eta_i^{\rm pc} \right) {\rm d}V = 1, \tag{69b}$$

$$\sum_{i=1}^m \boldsymbol{\varepsilon}_i^{\rm pc} = \nabla^{\rm s} \mathbf{u}^c \text{ in } V, \quad \mathbf{u}^c = \mathbf{0} \text{ on } S_u \tag{69c}$$

$$\sum_{i=1}^m \boldsymbol{\xi}_i^c = \mathbf{0} \text{ in } V. \tag{69d}$$

This problem amounts to minimizing the ratio on the r.h. side of Eq. (56) with respect to the subset of PAMs in M^+. It can be easily recognized that the Euler–Lagrange equations of the above problem coincide with those of Eqs. (60a), (60b) and (60c) and that Eqs. (60a), (60b) and (60c) and Eqs. (69a), (69b), (69c) and (69d) are the dual of each other, but this point is not further elaborated for simplicity. Considering that D is not differentiable for zero values of its arguments, there exists a computational convenience in substituting Eqs. (69a), (69b), (69c) and (69d) with the following one:

$$\beta_{\rm sh}^{-1} = \max_{\left(\varepsilon_i^{\rm pc},\, \xi_i^c,\, \eta_i^{\rm pc},\, u^c\right)} \int_V \sum_{i=1}^m \left(\bar{\boldsymbol{\sigma}}_i^E:\boldsymbol{\varepsilon}_i^{\rm pc} + \bar{\theta}_i \eta_i^{\rm pc} \right) {\rm d}V, \tag{70a}$$

subject to:

$$\int_V \sum_{i=1}^m D\left(\boldsymbol{\varepsilon}_i^{\rm pc}, \boldsymbol{\xi}_i^c, \eta_i^{\rm pc}\right) {\rm d}V - \int_V \left[\boldsymbol{\sigma}^{EP}:\Delta\boldsymbol{\varepsilon}^{\rm pc} + \theta_0 \Delta\eta^{\rm pc}\right] {\rm d}V = 1 \text{ plus Eqs. (69c) and (69d)} \tag{70b}$$

see, e.g. Pycko and Mróz, 1992; Stumpf and Le, 1991.

The type of collapse mode exhibited by the impending inadaptation collapse of Eqs. (63c), (63d), (63e) and (63f) depends, for a given structure, on several factors, namely:

1. The shape of the elastic stress domain, Π_σ, i.e. the polyhedral domain of the stress space, which is specified by the stress points $\bar{\boldsymbol{\sigma}}_i^E(x)$, $\forall i \in I_\sigma \subseteq I(m)$, at points $\mathbf{x} \in V$.
2. The hardening law in the thermo-plastic constitutive equations.
3. The hardening potential, $\Psi(\boldsymbol{\xi})$, or also the hardening matrix $\mathbf{H}(\boldsymbol{\xi})$.

In order to briefly discuss this point, let us consider the sum on the l.h. sides of Eqs. (63d) and (63e) and let them be respectively denoted with $\Delta\boldsymbol{\varepsilon}^{\rm pc}$ and $\Delta\boldsymbol{\xi}^c$. Using the plastic flow laws in Eq. (63c), we can write:

$$\Delta \boldsymbol{\varepsilon}^{pc} = \sum_{i=1}^{m} \lambda_i^c \frac{\partial f}{\partial \hat{\boldsymbol{\sigma}}_i} \left(\hat{\boldsymbol{\sigma}}_i, \hat{\boldsymbol{\chi}}, \hat{\theta}_i \right) \text{ in } V \tag{71a}$$

and

$$-\Delta \boldsymbol{\xi}^{c} = \sum_{i=1}^{m} \lambda_i^c \frac{\partial f}{\partial \hat{\boldsymbol{\chi}}_i} \left(\hat{\boldsymbol{\sigma}}_i, \hat{\boldsymbol{\chi}}, \hat{\theta}_i \right) = \mathbf{0} \text{ in } V. \tag{71b}$$

In the case of *perfect plasticity*, only Eq. (71a) is meaningful and both collapse modes are possible in principle, depending on the shape of the elastic stress response domain Π_σ (Polizzotto, 1993). For kinematically hardening materials, it is $\Delta\boldsymbol{\xi}^c = \mathbf{0}$, $\Delta\boldsymbol{\varepsilon}^{pc} = \mathbf{0}$, i.e. the impending collapse mode is always alternating plasticity. For isotropically hardening materials, there is a pair of internal scalar variables, say χ_{iso}, ξ_{iso}, with $\partial f/\partial\chi_{iso} = -\partial\kappa/\partial\chi_{iso} < 0$ and thus, the condition $\Delta\xi_{iso}^c = 0$ can be satisfied if, and only if, all λ_i^c coefficients are identically vanishing, i.e. the limit state is never reached and $\beta_{sh} = \infty$ (Polizzotto et al., 1991).

Finally, for hardening materials characterized by hardening matrix $\mathbf{H}(\boldsymbol{\xi})$ tending to vanish for $||\boldsymbol{\xi}|| \to \xi_{lim}$, the dual internal variables tend to become constant with increasing $||\boldsymbol{\xi}||$, i.e. $||\boldsymbol{\chi}(\boldsymbol{\xi})|| \to \chi_{lim}$, hence $\partial f/\partial\boldsymbol{\chi} \to \mathbf{0}$, for $||\boldsymbol{\xi}|| \to \xi_{lim}$; thus, $\Delta\boldsymbol{\xi}^c$ in Eq. (71b) tends to vanish too, without the vanishing of the λ_i^c's, such that $\boldsymbol{\Delta}\varepsilon^{pc}$ is then allowed to take vanishing or nonvanishing values, and both types of collapse modes are possible in principle. The latter result may also be achieved by introducing a saturation surface as in Fuschi and Polizzotto (1998), and Fuschi (1998), but this point is not further discussed here for brevity.

9. Applications

9.1. Thermoplastic von Mises material models in perfect plasticity

Let us consider the temperature-dependent von Mises yield function of the form

$$f = \sqrt{3J_2} - \kappa_0(1 - c\theta^2) \leq 0, \tag{72}$$

where J_2 is the second invariant of the deviatoric stress, $\boldsymbol{\sigma}'$, (i.e. $J_2 = 1/2\boldsymbol{\sigma}' : \boldsymbol{\sigma}'$), $c > 0$ and κ_0 are given constants. For any assigned deformation mechanism, ($\dot{\boldsymbol{\varepsilon}}^p$, $\dot{\eta}^p$), by Eqs. (8a), (8b) and (9), we obtain — dropping the procedural details for brevity — the relevant thermoplastic dissipation function as

$$D = \kappa_0 \dot{\bar{\varepsilon}}^p + \frac{1}{4c\kappa_0\dot{\bar{\varepsilon}}^p} \dot{\eta}^{p2}, \tag{73}$$

where, by definition,

$$\dot{\bar{\varepsilon}}^p = \sqrt{\frac{2}{3} \dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p}. \tag{74}$$

The above equations can be particularized to the case of one-dimensional stress state, obtaining:

$$f = |\sigma| - \kappa_0(1 - c\theta^2) \leq 0 \tag{75}$$

and

$$D = \kappa_0 |\dot{\varepsilon}^{\rm p}| + \frac{\dot{\eta}^{\rm p2}}{4c\kappa_0 |\dot{\varepsilon}^{\rm p}|}. \tag{76}$$

9.2. Two-bar system subjected to cycles of temperature variation

The system of Fig. 1(a) is composed of two elastic, perfectly plastic bars and a lower rigid block. Bar #1, of cross section A, is maintained at constant temperature $\theta_{(1)} = \theta_0$; whereas bar #2, of cross section $2A$ is subjected to cycles of temperature variations, $\theta_{(2)} = \theta_0 + \bar{\theta}\mu$ $(0 \le \mu \le 1)$, as shown in Fig. 1(b). A steady load $P = \alpha\bar{P}$ is applied on the rigid block, where $\bar{P} = 3A\kappa_0$ is the plastic collapse load and κ_0 is the yield stress at temperature $\theta = 0$. The material model is described by Eqs. (75) and (76). With the definitions of Section 5, Π^L is empty, whereas Π^θ coincides with the segment $0 \le \mu \le 1$, and μ turns out to be the single component of vector $\mathbf{Q}^\theta$, i.e. $Q_1^\theta \equiv \mu$. There are, thus, two dominant temperature variations, i.e.

$$\theta_{(1)1} - \theta_0 = 0, \quad \theta_{(2)1} - \theta_0 = \bar{\theta}\beta = \frac{\kappa_0}{aE}\beta \quad \text{for } \mu = 1 \tag{77a}$$

and

$$\theta_{(1)2} - \theta_0 = \theta_{(2)2} - \theta_0 = 0 \quad \text{for } \mu = 0, \tag{77b}$$

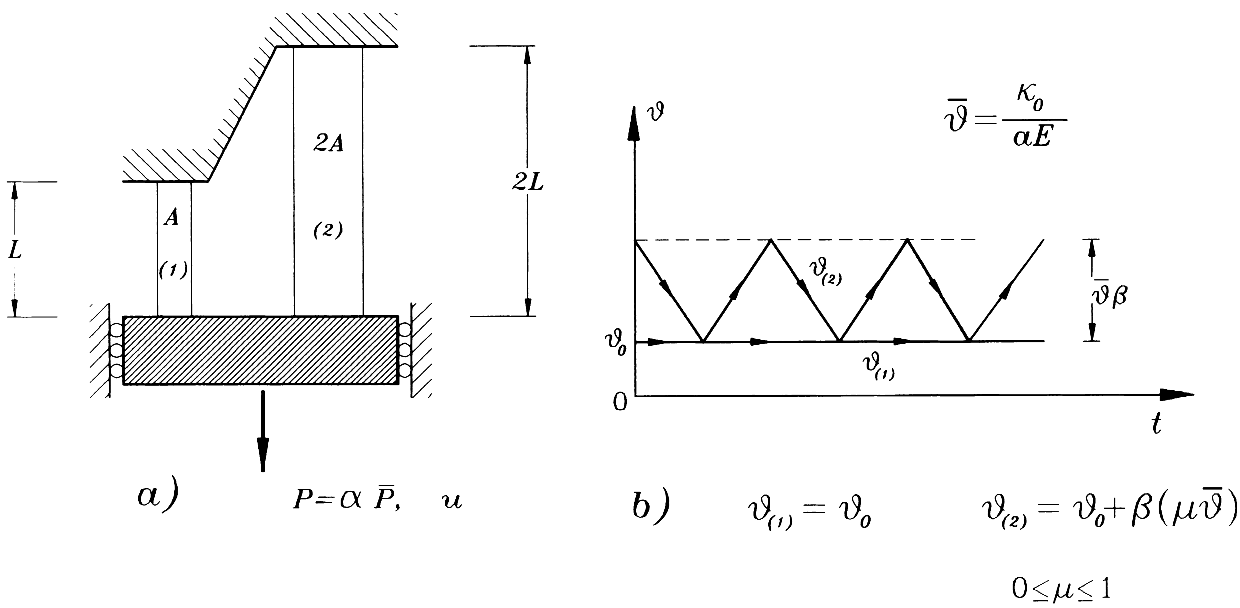

Fig. 1. Two-bar system subjected to a fixed mechanical load and to cyclically varying thermal loading: (a) Geometrical and loading scheme. (b) Temperature history in the two bars.

where $\bar{\theta} = \kappa_0/aE$ is assumed to be the reference temperature variations, a is the thermal expansion coefficient (by hypothesis, temperature independent) and E is Young modulus. The elastic stress responses to the dominant thermal loads for $\beta = 1$ are

$$\bar{\sigma}^E_{(1)1} = Ea\bar{\theta} = \kappa_0, \quad \bar{\sigma}^E_{(2)1} = -\frac{\kappa_0}{2} \quad \text{for } \mu = 1 \tag{78a}$$

and

$$\bar{\sigma}^E_{(1)2} = \bar{\sigma}^E_{(2)2} = 0; \quad \text{for } \mu = 0 \tag{78b}$$

whereas the elastic stress response to the steady load $\bar{P}$ is: $\bar{\sigma}^{EP}_{(1)} = 3\kappa_0/2$ and $\bar{\sigma}^{EP}_{(2)} = \bar{\sigma}^{EP}_{(1)}/2$. Also, the yield stress turns out to be constant in bar #1, i.e.

$$\sigma_{y(1)} = \sigma_{y0} = \kappa_0 c_0, \quad c_0 \equiv 1 - c\theta_0^2 \leq 1, \tag{79}$$

where σ_{y0} denotes the yield stress at temperature θ_0, but it is variable in bar #2, i.e.

$$\sigma_{y0} \geq \sigma_{y(2)} \geq \sigma_{y*}, \tag{80}$$

where σ_{y*} denotes the yield stress at the maximum temperature, $\theta_0 + \bar{\theta}\beta$, i.e.

$$\sigma_{y*} = \kappa_0\left[1 - c\left(\theta_0 + \bar{\theta}\beta\right)^2\right] = \kappa_0\left(c_0 - 2c_1\beta - c_2\beta^2\right), \tag{81}$$

where, by definition,

$$c_1 = c\bar{\theta}\theta_0, \quad c_2 = c\bar{\theta}^2. \tag{82}$$

The interaction Bree-like diagram of the (α, β)-plane is shown in Fig. 2(a), where B_S is the (elastic) shakedown > zone, B_F the reverse plasticity (or plastic shakedown) zone, and B_R the ratchetting zone. This diagram has been obtained by the so-called direct method (Polizzotto, 1994), illustrated by Fig. 2(b–c). Note that, on superposition of the self-stresses $\rho_{(1)} = \kappa_0\beta/2$ and $\rho_{(2)} = -\rho_{(1)}$ to the elastic stresses $\sigma^E_{(1)} = \kappa_0\beta\mu$ and $\sigma^E_{(2)} = \sigma^E_{(1)}/2$, respectively, one obtains the so-called post-transient stresses, i.e. $\hat{\sigma}_{(1)} = \kappa_0\beta(\mu - 1/2)$ and $\hat{\sigma}_{(2)} = -\sigma_{(1)}/2$. These describe the elastic stress paths $\hat{S}_{(1)}$ and $\hat{S}_{(2)}$, respectively, both of which are in a neutral configuration, i.e. symmetrically located with respect to the stress origin, Fig. 2(b,e). For $\beta < 2c_0$, one has to consider the stress state of Fig. 2(b), where both bars possess the reduced plastic resistance $\kappa_0 c_0$ for positive axial forces and, respectively, $\kappa_0 c_0$ and $\kappa_0(c_0 - 2c_1\beta - c2\beta^2)$ for negative axial forces. Thus, the plastic collapse load of the system turns out to be:

$$\alpha\bar{P} = A\kappa_0\left(c_0 - \frac{\beta}{2}\right) + 2A\kappa_0\left(c_0 - \frac{\beta}{4}\right) \quad \text{for } \alpha > 0 \tag{83a}$$

and

$$-\alpha\bar{P} = A\kappa_0\left(c_0 - \frac{\beta}{2}\right) + 2A\kappa_0\left(c_0 - 2c_1\beta - c_2\beta^2 - \frac{\beta}{4}\right) \quad \text{for } \alpha < 0 \tag{83b}$$

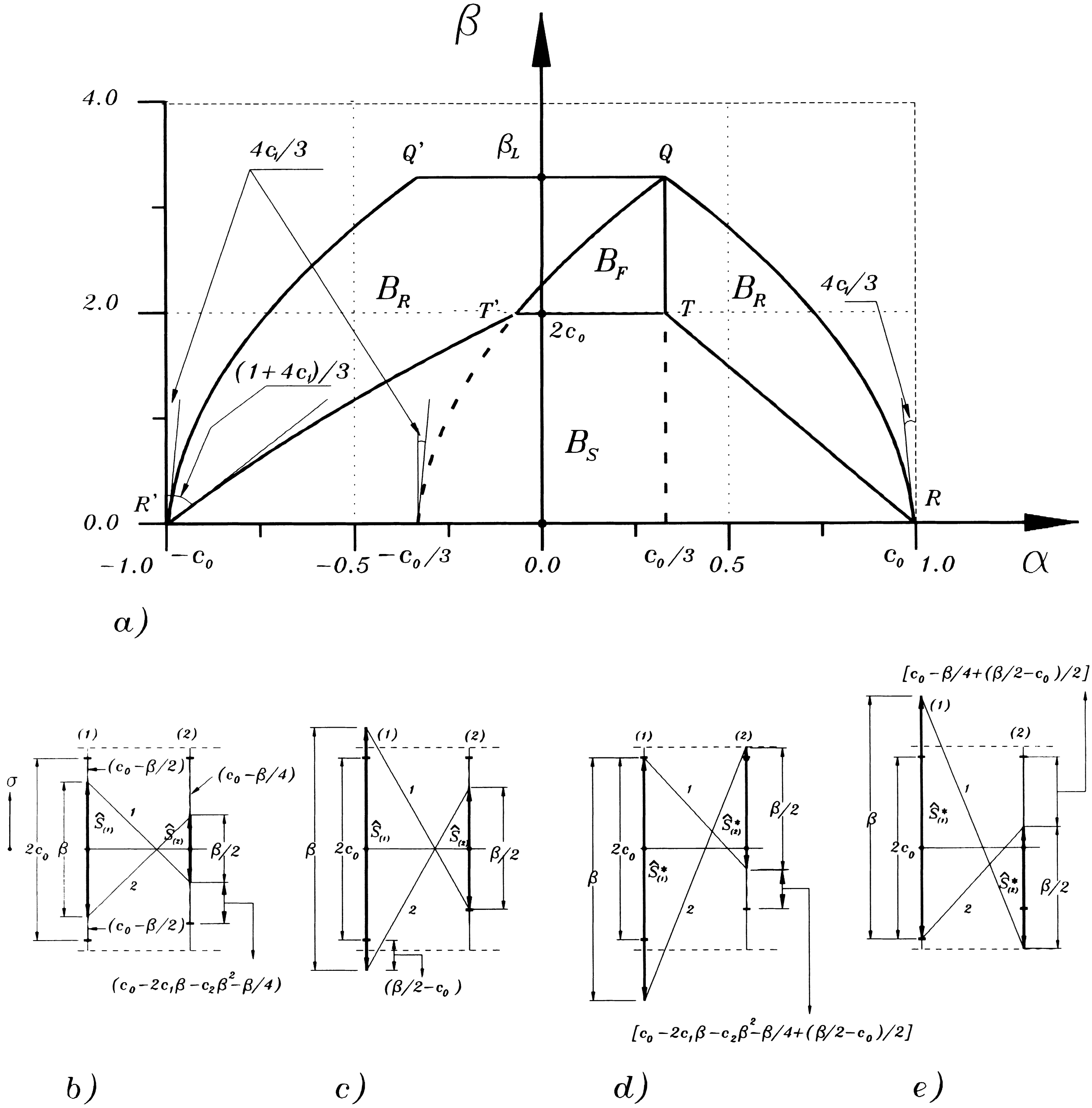

Fig. 2. Steady response of the two-bar system of Fig. 1. (a) Bree-like interaction diagram: Elastic shakedown domain, $B_S = \{RTT'R'\}$; Alternating plasticity domain, $B_F = \{TT'Q\}$; Ratchetting domain, $B_R = \{RTQ\}$ and $\{R'T'QQ'\}$. (b–e) Stress paths for the direct location of the shakedown domain.

from where, since $\bar{P} = 3\kappa_0 A$, one obtains the equations

$$\alpha = c_0 - \frac{\beta}{3} \quad (\alpha > 0) \tag{84a}$$

and

$$-\alpha = c_0 - (1 + 4c_1)\frac{\beta}{3} - 2c_2\frac{\beta^2}{3} \quad (\alpha < 0) \tag{84b}$$

which are respectively represented by lines RT and $R'T'$ in Fig. 2(a). The zone $RTT'R'R$ is the B_S zone.

For $\beta > 2c_0$, one has to consider the stress state sketched in Fig. 2(c), where the resistance of bar #1 has been exceeded. For $\mu = 1$, bar #1, works in traction at its reduced limit $\kappa_0 c_0$, whereas bar #2 works in compression below its reduced negative limit $+\kappa_0(c_0 - 2c_1\beta - c_2\beta^2)$. Thus, moving the elastic stress path $\hat{S}_{(1)}$ downwards to $\hat{S}^*_{(1)}$ through a shift $(\beta/2 - c_0)$, such that the upper end of $\hat{S}^*_{(1)}$ is at the positive reduced limit $\kappa_0 c_0$, Fig. 2(d), the elastic stress path $\hat{S}_{(2)}$ will correspondingly move upward by $(\beta/2 - c_0)/2$ and bar #2 will exhibit a remaining negative plastic resistance as $\kappa_0[(c_0 - 2c_1\beta - c_2\beta^2) + (\beta/2 - c_0)/2]$. Then, the plastic collapse load of the system is

$$-\alpha\bar{P} = 2A\kappa_0\left[c_0 - 2c_1\beta - c_2\beta^2 + \left(\frac{\beta}{2} - c_0\right)/2 - \frac{\beta}{4}\right], \tag{85}$$

from which the equation follows

$$-\alpha = \frac{c_0}{3} - 4c_1\frac{\beta}{3} - 2c_2\frac{\beta^2}{3}, \tag{86}$$

which is represented by the line $T'Q$ of Fig. 2(a). Analogously, for $\mu = 0$, bar #1 works in compression at its negative reduced limit $\kappa_0 c_0$, whereas bar #2 works in traction below its positive reduced limit $\kappa_0 c_0$. Thus, moving the elastic stress path $\hat{S}_{(1)}$ upwards to $\hat{S}^*_{(1)}$, Fig. 2(e), and $\hat{S}_{(2)}$ downwards to $\hat{S}^*_{(2)}$, the positive plastic collapse load can be computed:

$$\alpha\bar{P} = 2A\kappa_0\left[c_0 + \left(\frac{\beta}{2} - c_0\right)/2 - \frac{\beta}{4}\right]; \tag{87}$$

where follows the equation

$$\alpha = \frac{c_0}{3} \tag{88}$$

which is represented by line TQ of Fig. 2(a). Point Q is the intersection point of Eqs. (88) and (90) and its location β_L is specified by the temperature variation amplitude for which $\sigma_{y*} = 0$, i.e.

$$\beta_L = \frac{1 - \sqrt{c}\theta_0}{\sqrt{c_2}} = \frac{\sqrt{1 - c_0} + c_0 - 1}{c_1}. \tag{89}$$

The diagram is bounded, from above, by the line $\beta = \beta_L$ and laterally by the plastic resistance curves, RQ and $R'Q'$, the equations of which are easily found to be

$$\pm\alpha = c_0 - 4c_1\frac{\beta}{3} - 2c_2\frac{\beta^2}{3}. \tag{90}$$

The area TQT' is the reverse plasticity (or plastic shakedown) zone B_F, whereas the ratchetting zone

B_R is represented by the areas $RTQR$ and $R'T'QQ'R'$. The shape of the diagram of Fig. 2(a) depends on the actual value of the material parameters c_0, c_1 and c_2; the values $c_0 = 0.995$, $c_1 = 0.02$ and $c_2 = 0.08$ have been adopted in Fig. 2(a) (obtained for $\theta_0 = 50°C$, $\bar{\theta} = 200°C$, $c = 2 \times 10^{-6}\kappa_0 = 16.55$ MPa). Note that the case $c_2 = 0$ implies that $\kappa_0 \to \infty$, $c_0 \to 0$, $\theta_0 \to \infty$, but $\kappa_0 c_0 = \sigma_{y0}$ and $2\kappa_0 c_1 \to K\sigma_{y0}$. Thus, Eq. (91) yields

$$\beta_L = \frac{\sqrt{\kappa_0 - \sigma_{y0}} + \sigma_{y0} - \kappa_0}{\kappa_0 c_1} \simeq \frac{\sigma_{y0}}{2\kappa_0 c_1} = \frac{1}{K}, \tag{91}$$

which coincides with the result given by Polizzotto (1994) for the case of yield stress being linearly dependent on the temperature variation.

Using Eq. (56), but modified according to Eqs. (65) and (63f), upper bound to β_{sh} (i.e. the shakedown boundary $RTT'R'$ of Fig. 2(a)) can be obtained using the kinematic theorem. With the notation, $D_{(k)i} = D(\varepsilon^{pc}_{(k)i}, \eta^{pc}_{(k)i})$, we can write the ratio:

$$\beta_1 \equiv \frac{D_{(1)1} + D_{(1)2} + 4D_{(2)1} + 4D_{(2)2} - 3\alpha\kappa_0 u/L - \theta_0\left(\Delta\eta^{pc}_{(1)} + 4\Delta\eta^{pc}_{(2)}\right)}{\kappa_0\varepsilon^{pc}_{(1)1} - 2\kappa_0\varepsilon^{pc}_{(2)1} + 4\bar{\theta}\eta^{pc}_{(2)1}}, \tag{92}$$

where

$$\varepsilon^{pc}_{(1)1} + \varepsilon^{pc}_{(1)2} = \frac{u}{L}$$

and

$$\varepsilon^{pc}_{(2)1} + \varepsilon^{pc}_{(2)2} = \frac{u}{2L}. \tag{93}$$

Considering a ratchetting collapse mode with $u > 0$, we have $\varepsilon^{pc}_{(1)2} = \varepsilon^{pc}_{(2)1} = 0$ and $\eta^{pc}_{(1)2} = \eta^{pc}_{(2)1} = 0$, such that Eq. (92), by Eq. (93), becomes

$$\begin{aligned}\beta_1 &= \frac{D_{(1)1} + 4D_{(2)2}}{\kappa_0 u/L} - 3\alpha - \frac{\theta_0\left(\eta^{pc}_{(1)1} + 4\eta^{pc}_{(2)2}\right)}{\kappa_0 u/L} \\ &= 3(1-\alpha) + \frac{\eta^{pc2}_{(1)1}}{4c\kappa_0^2(u/L)^2} + \frac{4\eta^{pc2}_{(2)2}}{4c\kappa_0^2(u/L)(u/2L)} - \frac{\theta_0\left(\eta^{pc}_{(1)1} + 4\eta^{pc}_{(2)2}\right)}{\kappa_0 u/L} \\ &= 3(c_0-\alpha) + c\left(\frac{\eta^{pc}_{(1)1}}{4c\kappa_0 u/L} - \theta_0\right)^2 + 2c\left(\frac{\eta^{pc}_{(2)2}}{4c\kappa_0\, u/2L} - \theta_0\right)^2\end{aligned} \tag{94}$$

and thus

$$\beta_1 \geq 3(c_0 - \alpha). \tag{95}$$

Considering an alternating plasticity mode i.e. $u = 0$, Eq. (92) reads:

$$\eta_2 \equiv \frac{2D_{(1)1} + 8D_{(2)1} - \theta_0\left(\Delta\eta^{pc}_{(1)} + \Delta\eta^{pc}_{(2)}\right)}{\kappa_0\varepsilon^{pc}_{(1)1} - 2\kappa_0\varepsilon^{pc}_{(2)1} + 4\bar{\theta}\eta^{pc}_{(2)1}}, \tag{96}$$

which takes a smaller value if $\varepsilon^{pc}_{(2)1} = \eta^{pc}_{(2)1} = 0$; that is, using Eq. (76) with $\varepsilon^{pc}_{(1)1} > 0$ and $\eta^{pc}_{(1)2} = \eta^{pc}_{(1)1}$, we have

$$\eta_2 = 2 + \frac{2\eta^{pc2}_{(1)1}}{4c\kappa_0^2\left(\varepsilon^{pc}_{(1)1}\right)^2} - \frac{2\theta_0\eta^{pc}_{(1)1}}{\kappa_0\varepsilon^{pc}_{(1)1}} = 2c_0 + 2c\theta_0^2 + \frac{2c\eta^{pc2}_{(1)1}}{\left(2c\kappa_0\varepsilon^{pc}_{(1)1}\right)^2} - \frac{4c\theta_0\eta^{pc}_{(1)1}}{2c\kappa_0\varepsilon^{pc}_{(1)1}}$$

$$= 2c_0 + 2c\left(\frac{\eta^{pc}_{(1)1}}{2c\kappa_0\varepsilon^{pc}_{(1)1}} - \theta_0\right)^2, \tag{97}$$

that is

$$\eta_2 \geq 2c_0. \tag{98}$$

10. Review and conclusion

The shakedown problem for materials with temperature dependent yield functions has been addressed in this paper with the purpose to remedy some deficiencies exhibited by the existing static and kinematic theorems when applied to the case of thermal loading (possibly combined with mechanical loading). This goal has been reached by assuming the yield function convex in the stress–temperature space and by restating the shakedown theorems within the framework of a more refined thermo-plasticity theory in which temperature and plastic entropy rate play the role of additional state and evolutive variables.

Grounded on the idea that (total) entropy is the sum of the reversible part (i.e. entering the state equations) and on irreversible (or plastic) part (i.e. entering the evolutive equations), thermodynamic arguments have been developed in the hypothesis of small strains and internal variables to provide a firm rational basis to the above thermo-plasticity theory. In particular, the second thermodynamics principle has led to the concept of intrinsic thermo-mechanical dissipation density, which is the sum of a mechanical part (attached to the plastic strain and internal variables rates), and of a thermal part (attached to the plastic entropy rate). A maximum intrinsic thermo-plastic dissipation theorem, extension of the classical one to the present context, has been shown to characterize the relevant thermo-plasticity flow laws.

In the hypothesis of negligible thermo-mechanical coupling effects, a polyhedral load/temperature domain has been considered in order to formulate the two shakedown theorems with suitable discrete forms in which the so-called dominant (or basic) load/temperature conditions are involved, what is rendered possible by the (full) convexity of the yield function. Both necessary and sufficient conditions have been proved for the two theorems, showing that both of them provide dual lower and upper bound statements for the shakedown safety factor. This is in contrast to the existing shakedown theorems, which are nevertheless recovered whenever there is a stationary temperature field to consider as a permanent loading combined with time-variable mechanical loads, since in fact in this case all terms containing plastic entropy drop from the extended formulation.

The static and kinematic approaches to the problem for the evaluation of the shakedown safety factor have been shown to save their classical formats, but the kinematic approach is based on plastic accumulation mechanisms that include — beside plastic strain and kinematic internal variable fields — plastic entropy fields. The shakedown limit state, i.e. the state of the structure caused by the shakedown limit load, has been shown to be characterized, among other, by the impending noninstantaneous plastic (or inadaptation) collapse mechanism which is about to establish in the structure under loadings slightly exceeding the shakedown limit.

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References

Boley, B.A., Weiner, J.H., 1960. Theory of Thermal Stresses. Wiley, New York.

Borino, G., Polizzotto, C., 1997a. Shakedown theorems for a class of materials with temperature-dependent yield stress. In: Owen, D.R.J., Oñate, E., Hinton, E. (Eds.), Computational Plasticity. CIMNE, Barcelona, Spain, pp. 475–480 Part 1.

Borino, G., Polizzotto, C., 1997b. Reformulation of shakedown theorems for materials with temperature dependent yield stress. In: Thermal Stresses '97, Proc. Second Int. Symp. on Thermal Stresses and related topics. Rochester Institute of Tecnology, Rochester, NY, pp. 89–92.

Drucker, D.C., 1960. Plasticity. In: Goodier, J.N., Hoff, J.H. (Eds.), Structural Mechanics. Pergamon Press, London, pp. 155–174.

Fuschi, P., 1999. Structural shakedown for elastic–plastic materials with hardening saturation surface. Int. J. Solids Struct. 36, 219–240.

Fuschi, P., Polizzotto, C., 1998. Internal-variable constitutive model for rate-independent plasticity with hardening saturation surface. Acta Mechanica 129, 73–95.

Germain, P., Nguyen, Q.S., Suquet, P., 1983. Continuum thermodynamics. J. Appl. Mech. ASME 50, 1010–1021.

Gokhfeld, D.A., Cherniavsky, D.F., 1980. Limit Analysis of Structures at Thermal Cycling. Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands.

Halphen, B., 1979. Steady cycles and shakedown in elastic–viscoplastic and plastic structures. In: Matériaux et Structures Sous Chargement Cyclique. Association Amicale des Ingeniers Anciens Eléves de E.N.P.C, Paris, pp. 203–230.

Hansen, N.R., Schreyer, A., 1994. A thermodynamically consistent framework for theories of elastoplasticity coupled with damage. Int. J. Solids Struct. 31, 359–389.

Hill, R., 1950. The Mathematical Theory of Plasticty. Oxford University Press, Oxford.

König, J.A., 1982a. On some recent developments in the shakedown theory. Advances in Mechanics 5 (1–2), 237–258.

König, J.A., 1982b. Shakedown criteria in the case of loading and temperature variations. J. de Mécanique Thérique et Appliquée (Special Issue), 99–108.

König, J.A., 1987. Shakedown Analysis of Elastic–Plastic Structures. PWN–Polish Scientific Publishers, Warsaw.

Koiter, W.T., 1960. General theorems of elastic–plastic solids. In: Sneddon, J.N., Hill, R. (Eds.), Progress in Solid Mechanics, vol. 1. North Holland, Amsterdam, pp. 167–221.

Lemaitre, J., Chaboche, J.-L., 1985. Mécanique des Matériaux Solides. Dunod, Paris.

Lubliner, J., 1990. Plasticity Theory. Macmillan Publishing Co, New York.

Maier, G., 1987. A generalization to nonlinear hardening of the first shakedown theorem for discrete elastic–plastic models. Atti Acc. Lincei Rend. Fis. 8 (LXXXI), 161–174.

Pantuso, D., Bathe, K.J., 1997. Finite element analysis of thermo-elastic–plastic solids in contact. In: Owen, D.R.J., Oñate, E., Hinton, E. (Eds.), Computational Plasticity. CIMNE, Barcelona, Spain, pp. 72–87 Part 1.

Panzeca, T., Polizzotto, C., 1988. On shakedown of elastic–plastic solids. Meccanica 23, 94–101.

Polizzotto, C., 1993. On the conditions to prevent plastic shakedown of structures. Part I and Part II. J. Appl. Mech. ASME 60, 15–19 (and pp. 20–25).

Polizzotto, C., 1994. Steady states and sensitivity analysis in elastic–plastic structures subjected to cyclic loads. Int. J. Solids Struct. 31, 953–970.

Polizzotto, C., 1995. Elastic–viscoplastic solids subjected to thermal and loading cycles. In: Mórz, Z., Weichert, D., Dorosz, S. (Eds.), Inelastic Behaviour of Structures under Variable Loads. Kluwer Academic Publishers, Dordrecht, The Netherlands, pp. 95–128.

Polizzotto, C., Borino, G., Caddemi, S., Fuschi, P., 1991. Shakedown problems for material models with internal variables. Eur. J Mech. A/Solids 10, 621–639.

Prager, W., 1956. Shakedown in elastic plastic media subjected to cycles of load and temperature. In: La Plasticità nella Scienza delle Costruzioni, Proc. A. Dannuso Symp. Zanichelli, Bologna, pp. 239–244 Varenna, Italy.

Pycko, S., Mróz, Z., 1992. Alternative approach to shakedown as a solution of a min–max problem. Acta Mechanica 93, 205–222.

Rozemblum, V.I., 1965. On analysis of shakedown of uneven heated plastic bodes. PMTF, 98–101.

Simo, J.C., Miehe, C., 1992. Associative thermoplasticity at finite strains: formulation, numerical analysis and implementation. Comp. Meths. Appl. Mech. Engng. 98, 41–104.

Stumpf, H., Le, K.C., 1991. On shakedown of elastoplastic shells. Q. Appl. Math. XLIX, 781–793.

Svedberg, T., Runesson, K., 1997. Gradient-regularized hyperelasto–plasticity coupled to damage — thermodynamics and a prototype model for uniaxial stress. In: Owen, D.R.J., Oñate, E., Hinton, E. (Eds.), Computational Plasticity. CIMNE, Barcelona, Spain, pp. 524–531 Part 1.